

On the stability of regular algebras

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Introduction

In [4], a notion of stability was introduced for non-commutative graded algebras (connected and generated in degree 1). Moreover, quasi-projective moduli stacks of Deligne-Mumford type were constructed, for stable graded algebras of fixed Hilbert series. Here we are concerned with the Hilbert series $(1 - t)^{-3}$, which is the Hilbert series of the projective plane.

Interesting examples of algebras with this Hilbert series are *quadratic regular algebras* of global dimension 3 (see [2], [3]). In this article, we determine exactly which of these algebras are stable, and then describe the moduli stack of stable regular algebras.

In [3], it was proved that all quadratic regular algebras of dimension 3 are associated to *elliptic triples* (X, σ, L) , consisting of a scheme X , an automorphism σ of X , and line bundle L , which embeds X into \mathbb{P}^2 as a divisor of degree 3. We prove that the regular algebra A determined by such a triple is stable if and only if X does not contain any linear component preserved by σ , and it does not contain any singularity of X fixed by σ . This leaves only two stable cases: the case where X is non-singular, and the case where X is a Neron triangle and the automorphism σ acts transitively on the set of its components/sides.

See Section 1.3 for a more detailed outline of the proof of this result, which is the main contribution of this paper.

The moduli stack of these stable algebras turns out to be smooth, and to decompose into 4 irreducible components, of dimensions 2, 1, 0, and 0, respectively, see Theorem 3.2. These components can be characterized by the order of the automorphism which σ induces on $\text{Pic}^0(X)$. This order can be 1, 2, 3, or 4. In the classification of [2], these components correspond to Types A, B, E, and H, respectively. Unfortunately, neither of the two positive-dimensional components is proper.

We prove that the property ‘stable *and* regular’ is an open property, for flat families of graded algebras. Therefore, the moduli stack we construct is open in the moduli stack of all stable algebras of Hilbert series $(1 - t)^{-3}$, and hence also dense in each component which it intersects. We do not address the question of whether or not there are components of the moduli stack of stable algebras which do not contain any regular algebras.

Let us emphasize that our goal here is not to classify quadratic regular algebras, or describe their moduli. This has been done elsewhere, see [3] for the classification, and [1], and references therein, for moduli. Rather, the purpose of this work is to determine how our notion of stability relates to these known moduli spaces.

The question of compactifying our moduli stack turns out to be rather subtle: the geometric invariant theory considerations of [4] provide us with compactifications using semi-stable algebras, but these compactifications depend on an integer q , because they are constructed in terms of algebras truncated beyond degree q . Preliminary considerations show that the boundaries of these compactifications grow, as q increases.

There is a natural absolute notion of semi-stability, but it does not give rise to compact moduli. We conjecture that all quadratic regular algebras of dimension 3 are semi-stable. Examples show that the converse is not true: there are semi-stable algebras which are degenerate, in the sense of [3]. On the other hand, we do not know any examples of stable algebras which are not regular. For details on semi-stability, see [6].

We will work over an algebraically closed field of characteristic avoiding 2 and 3, and call it \mathbb{C} . All our graded algebras A will be connected, i.e.,

$$A = \bigoplus_{n \geq 0} A_n$$

with $A_0 = \mathbb{C}$, and generated in degree 1.

1 Stability of regular algebras

We start by reviewing the notion of stability for graded algebras from [4], to set up notation. Then we review the theory of quadratic regular algebras of dimension 3, as far as it is relevant for us. We announce our main result, characterizing the stable regular algebras, and outline the proof.

We prove some general facts that we will later need in the proof.

1.1 Stability

Let A be a connected graded algebra, finitely generated in degree 1. We recall some notions from [4].

Definition 1.1 The algebra A is **q -stable**, (for an integer $q > 1$), if for every non-trivial test configuration for A , generated in degree 1, the Futaki function satisfies $F(q) > F(1)$. It is **stable**, if there exists an $N > 0$, such that it is q -stable for all $q > N$.

The degree 1 part of a test configuration is a filtration

$$A_1 = W^{(0)} \supset W^{(1)} \supset \dots, \quad (1)$$

(the inclusions are not strict), such that $W^{(k)} = 0$, for sufficiently large k . The *test configuration* $B \subset A[t, t^{-1}]$ generated by the filtration (1) is the $\mathbb{C}[t]$ -algebra defined by

$$B_n = \bigoplus_k t^{-k} \sum_{k_1 + \dots + k_n = k} W^{(k_1)} \dots W^{(k_n)}. \quad (2)$$

Here, for fixed $n > 0$, and $k \in \mathbb{Z}$, the sum is over all n -tuples of non-negative numbers (k_1, \dots, k_n) , such that $k_1 + \dots + k_n = k$. If $k < 0$, the set of such n -tuples is empty, and the coefficient of t^{-k} is $A_n = (A_1)^n$, by definition.

Setting

$$I_n^{(k)} = \sum_{k_1 + \dots + k_n = k} W^{(k_1)} \dots W^{(k_n)}$$

defines a descending sequence of two-sided graded ideals $A \supset I^{(1)} \subset I^{(2)} \dots$ in A , satisfying $I^{(k)} I^{(\ell)} \subset I^{(k+\ell)}$, for all $k, \ell \geq 0$. The fibre over $t = 0$ of B is the doubly graded algebra

$$B/tB = \bigoplus_{k \geq 0} I^{(k)} / I^{(k+1)}.$$

The *weight function* of the test configuration (2) is given by

$$w(n) = \sum_{k > 0} \dim I_n^{(k)} = \sum_{k > 0} \dim \left(\sum_{k_1 + \dots + k_n = k} W^{(k_1)} \dots W^{(k_n)} \right).$$

The weight $w(n)$ is equal to the total weight of the graded vector space $(B/tB)_n = \bigoplus_{k \geq 0} (B/tB)_n^{(k)}$:

$$w(n) = \sum_{k \geq 0} k \dim (B/tB)_n^{(k)}.$$

The *Futaki function* of the test configuration (2) is

$$F(n) = \frac{w(n)}{n \dim A_n}.$$

In particular,

$$F(1) = \frac{1}{\dim A_1} \sum_{k > 0} \dim W^{(k)}.$$

To test for stability, we can restrict to test configurations generated by filtrations such that $W^{(1)} \neq A_1$. We will always restrict attention to such test configurations.

1.2 Quadratic regular algebras of dimension 3

Recall ([3], Definition 4.5) that an **elliptic triple** is a triple (X, σ, L) , consisting of a scheme X , together with a very ample line bundle L , such that $V = \Gamma(X, L)$ is of dimension 3, and the closed immersion $X \rightarrow \mathbb{P}(V)$ makes X into a divisor of degree 3 inside $\mathbb{P}(V)$. Moreover, $\sigma : X \rightarrow X$ is an automorphism of X .

We will assume our triples to be **regular**, which means that

$$\sigma^* \sigma^* L \otimes L \cong \sigma^* L \otimes \sigma^* L,$$

but not **linear**, which is the stronger condition

$$\sigma^* L \cong L.$$

The quadratic algebra A associated to the regular elliptic triple (X, σ, L) is the quotient of the tensor algebra of V , by the two-sided ideal generated by the kernel R of

$$\begin{aligned} V \otimes V &\longrightarrow \Gamma(X, L \otimes \sigma^* L) \\ x \otimes y &\longmapsto x \otimes \sigma^* y. \end{aligned} \quad (3)$$

As (3) is surjective, and $\dim \Gamma(X, L \otimes \sigma^* L) = 6$, the kernel R is of dimension 3. Therefore, A has 3 generators in degree 1, and 3 relations in degree 2.

By [3] Theorem 6.8.(ii), the algebra A is a *regular algebra of global dimension 3*, in particular, its Hilbert series is given by $(1 - t)^{-3}$, equivalently, $\dim A_n = \frac{1}{2}(n+1)(n+2)$, for all $n \geq 0$.

The *twisted coordinate ring* B , associated to the triple (X, σ, L) is defined by

$$B_n = \Gamma(X, L_n),$$

where

$$L_n = L \otimes \sigma^* L \otimes \dots \otimes (\sigma^{n-1})^* L,$$

which multiplication given by

$$(x_0 \otimes \dots \otimes x_{n-1}) \cdot (y_0 \otimes \dots \otimes y_{m-1}) = x_0 \otimes \dots \otimes x_{n-1} \otimes (\sigma^n)^* y_0 \otimes \dots \otimes (\sigma^n)^* y_{m-1}.$$

We have $\dim B_n = 3n$, for $n > 0$.

There is a canonical morphism of graded \mathbb{C} -algebras

$$A \longrightarrow B, \quad (4)$$

induced by the identification $V = A_1 = B_1$. By [3] Theorem 6.8, the morphism (4) is an epimorphism, whose kernel is equal to both $c_3 A$ and $A c_3$, for an element $c_3 \in A_3$, which is both a left and a right non-zero divisor (of course, c_3 is unique up to multiplication by a non-zero scalar).

1.3 The main theorem

Theorem 1.2 (Stability of regular algebras) *Let (X, σ, L) be a regular elliptic triple, with associated regular algebra A . The following are equivalent:*

- (i) A is stable,
- (ii) A is q -stable, for every $q \geq 3$,
- (iii) A is 3-stable,
- (iv) X contains no linear component preserved by σ , and no singularity preserved by σ .

(In (iv), it is not required that the linear component be fixed pointwise by σ .)

An elliptic triple without linear components preserved by σ , and without singularities preserved by σ , is necessarily either smooth, or the union of a line and a smooth conic intersecting at two points, or a union of three lines

intersecting transversally at three nodes. The second of these three cases is not regular, but *exceptional*, see [3], 4.9, and is therefore excluded in the theorem.

Thus, if X is smooth, or a triangle on which σ acts by cyclic permutation of the edges, A is stable, in all other cases A is at best semi-stable.

Method of proof

To prove the theorem, we prove the following three propositions.

Proposition 1.3 *Let $U \subset V$ be a one-dimensional subspace, such that the line $Y = Z(U) \subset \mathbb{P}(V)$ is contained in $X \subset \mathbb{P}(V)$, and such that $\sigma|_Y : Y \rightarrow X$ factors through $Y \subset X$. Then the test configuration for A generated by the flag $V \supset U \supset 0$ of A_1 has constant Futaki function \tilde{F} , i.e., $\tilde{F}(n) = \tilde{F}(1)$, for all $n \geq 2$.*

Proposition 1.4 *Let $W \subset V$ be a 2-dimensional subspace, such that the point $P = Z(W) \in \mathbb{P}(V)$ is contained in X , is a singular point of X , and satisfies $\sigma(P) = P$. Then the test configuration for A generated by the flag $V \supset W \supset 0$ of A_1 has constant Futaki function.*

Proposition 1.5 (Stability estimates) *Consider the filtration*

$$V \supset \underbrace{W \supset \dots \supset W}_{\ell \text{ times}} \supset \underbrace{U \supset \dots \supset U}_{m \text{ times}} \supset 0 \quad (5)$$

of A_1 , where at least one of the two integers $\ell \geq 0$, $m \geq 0$ is positive, and $\dim U = 1$, $\dim W = 2$. Let $P = Z(W)$, and $Y = Z(U)$.

Suppose that if $P \in X$, and $\sigma(P) = P$, then P is a non-singular point of X , and assume that if $Y \subset X$, then X is a Neron triangle, whose sides are permuted cyclically by σ . Then for the Futaki function $F(n)$ of the test configuration generated by the flag (5), we have

$$\begin{cases} F(n) \geq F(1), & \text{for } n = 1, 2 \\ F(n) > F(1), & \text{for } n \geq 3. \end{cases}$$

Propositions 1.3 and 1.4 will be proved in the next section on twisted coordinate rings.

The stability estimates

Let us set up notation used in the proof of Proposition 1.5. Let (X, σ, L) be a regular elliptic triple, with associated regular algebra A and twisted coordinate ring $B = A/c_3 A$. The flag (5) generates test configurations $(I_n^{(k)})$ for B and $(J_n^{(k)})$ for A . Let $w(n)$ be the weight function of $(I_n^{(k)})$ and $\tilde{w}(n)$ the weight function of $(J_n^{(k)})$. Let $a(n) = \dim A_n$, and $b(n) = \dim B_n$, so

$$a(n) = \frac{1}{2}(n+1)(n+2), \quad \text{and} \quad b(n) = 3n.$$

For future reference, let us also introduce

$$\tilde{a}(n) = a(n-3) = \frac{1}{2}(n-1)(n-2).$$

Let $F(n)$ be the Futaki function of $(I_n^{(k)})$ and $\tilde{F}(n)$ the Futaki function of $(J_n^{(k)})$.

We have

$$\tilde{F}(n) = \frac{\tilde{w}(n)}{\frac{1}{2}n(n+1)(n+2)},$$

and

$$\tilde{F}(1) = F(1) = \frac{2\ell + m}{3},$$

because $\tilde{w}(1) = w(1) = 2\ell + m$.

For $n = 2$, we have $\tilde{w}(2) = w(2)$, and our claim that $\tilde{F}(2) \geq \tilde{F}(1)$, amounts to

$$w(2) \geq 8\ell + 4m.$$

For $n \geq 3$ our claim that $\tilde{F}(n) > \tilde{F}(1)$ amounts to

$$\tilde{w}(n) > \frac{2\ell + m}{6} n(n+1)(n+2). \quad (6)$$

To prove (6), we first prove that lower estimates for $w(n)$, imply lower estimates for $\tilde{w}(n)$ via a bootstrapping method using the short exact sequence

$$0 \longrightarrow A_{n-3} \xrightarrow{c_3} A_n \longrightarrow B_n \longrightarrow 0.$$

The larger the integer p such that $c_3 \in J_3^{(p)}$, the better the estimates are that the bootstrapping method gives. For different cases, different values for p apply. In many cases, it will be sufficient that $c_3 \in J_3^{(2\ell)}$, but in the case where Y is a linear component of X , we will need $c_3 \in J_3^{(3\ell)}$. We will prove these facts in Theorem 1.29 and Proposition 1.30.

Finally, the estimates for $w(n)$ are postponed to the next section under the heading ‘Estimates’.

Our estimates for $w(n)$ and $\tilde{w}(n)$ will always be of the form

$$w(n) \geq \ell w_\ell(n) + m w_m(n) \quad \text{and} \quad \tilde{w}(n) \geq \ell \tilde{w}_\ell(n) + m \tilde{w}_m(n), \quad (7)$$

with functions $w_\ell(n), w_m(n), \tilde{w}_\ell(n), \tilde{w}_m(n)$, which do not depend on ℓ or m .

The stability condition is

$$3\tilde{w}(n) - (2\ell + m)n a(n) > 0,$$

which we can deduce from

$$\ell(3\tilde{w}_\ell(n) - 2n a(n)) + m(3\tilde{w}_m(n) - n a(n)) > 0.$$

Let us therefore introduce notation

$$G_\ell(n) = 3\tilde{w}_\ell(n) - 2n a(n) = 3\tilde{w}_\ell(n) - n(n+1)(n+2),$$

and

$$G_m(n) = 3\tilde{w}_m(n) - n a(n) = 3\tilde{w}_m(n) - \frac{1}{2}n(n+1)(n+2).$$

With this notation, stability will be implied by

$$\ell G_\ell(n) + m G_m(n) > 0. \quad (8)$$

Definition 1.6 The one-dimensional subspace $U \subset V$ corresponding to the line $Z = Z(U) \subset \mathbb{P}(V)$ is called **special**, if Z is contained in X , i.e., if Z is a component of X .

In the non-special case, the zero locus of U in X is an effective Cartier divisor D on X , such that $L = \mathcal{O}(D)$.

Definition 1.7 The two-dimensional subspace $W \subset V$ corresponding to the point $P = Z(W) \in \mathbb{P}(V)$ is called **special**, if P lies on the smooth part of X and $\sigma(P) = P$.

One of the most useful facts about the non-special case is that it implies $VW + WV = B_2$, as we will see later.

If W is special, then P is an effective Weil divisor on X , and we have $W = \Gamma(L(-P))$.

Bootstrapping: comparing A and B using c_3

Proposition 1.8 Suppose that $c_3 \in J_3^{(p)}$. Then we have

$$\tilde{w}(n) \geq \sum_{1 \leq n-3i \leq n} (w(n-3i) + p a(n-3i-3)),$$

where the sum is over all integers i satisfying $1 \leq n-3i \leq n$.

PROOF. By properties of test configurations, we have $c_3 J_{n-3}^{(k)} \subset J_n^{(k+p)}$, for all $k \geq 0$ and all $n \in \mathbb{Z}$. Moreover, we have a surjection $J_n^{(k+p)} \twoheadrightarrow I_n^{(k+p)}$, whose kernel contains $c_3 J_{n-3}^{(k)}$, which proves that

$$\dim J_n^{(k+p)} \geq \dim I_n^{(k+p)} + \dim c_3 J_{n-3}^{(k)},$$

for all $k \geq 0$ and all $n \in \mathbb{Z}$. As c_3 is a non-zero divisor, we have $\dim c_3 J_{n-3}^{(k)} = \dim J_{n-3}^{(k)}$, and so

$$\dim J_n^{(k+p)} \geq \dim I_n^{(k+p)} + \dim J_{n-3}^{(k)}.$$

Thus we have

$$\begin{aligned}
\tilde{w}(n) &= \sum_{k=1}^{\infty} \dim J_n^{(k)} \\
&\geq \sum_{k=1}^{\infty} \dim I_n^{(k)} + p \dim A_{n-3} + \sum_{k=1}^{\infty} \dim J_{n-3}^{(k)} \\
&= w(n) + p a(n-3) + \tilde{w}(n-3),
\end{aligned}$$

because for $k \leq 0$, we have $J_{n-3}^{(k)} = A_{n-3}$. From this recursion, we deduce

$$\tilde{w}(n) \geq \sum_{i=0}^{\infty} w(n-3i) + p \sum_{i=1}^{\infty} a(n-3i),$$

which is our claim. \square

Corollary 1.9 *Assume that $c_3 \in J_3^{(q\ell+sm)}$. Then*

$$\tilde{w}(n) \geq \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} (w + (q\ell + sm)\tilde{a})(n-3i),$$

and hence we may use

$$\tilde{w}_\ell(n) = \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} (w_\ell + q\tilde{a})(n-3i)$$

and

$$\tilde{w}_m(n) = \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} (w_m + s\tilde{a})(n-3i).$$

Both $w_\ell(n)$ and $w_m(n)$ will turn out to be polynomial functions of degree 2 in n . So we will need the following sums later on.

Lemma 1.10

$$\begin{aligned}
\zeta_2(n) &= 3 \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} (n-3i)^2 = \frac{1}{6} n(n+3)(2n+3) + \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3} \\ -\frac{1}{3} & \text{if } n \equiv 1 \pmod{3} \\ +\frac{1}{3} & \text{if } n \equiv 2 \pmod{3}, \end{cases} \\
\zeta_1(n) &= 3 \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} (n-3i) = \frac{1}{2} n(n+3) + \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3} \\ +1 & \text{if } n \equiv 1 \pmod{3} \\ +1 & \text{if } n \equiv 2 \pmod{3}, \end{cases} \\
\zeta_0(n) &= 3 \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} 1 = n + \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3} \\ +2 & \text{if } n \equiv 1 \pmod{3} \\ +1 & \text{if } n \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

Remark 1.11 Note that, no matter the divisibility of n modulo 3,

$$n(n+1)(n+2) = 3\zeta_2(n) - 3\zeta_1(n) + 2\zeta_0(n). \quad (9)$$

Because of this, it will turn out to be convenient to express all our estimates for $G_\ell(n)$ and $G_m(n)$ in terms of $\zeta_2(n)$, $\zeta_1(n)$ and $\zeta_0(n)$.

1.4 Twisted homogeneous coordinate rings

Suppose (X, σ, L) is an elliptic triple, with associated twisted homogeneous coordinate ring

$$B = \bigoplus_n \Gamma(X, L_n), \quad L_n = L \otimes L^\sigma \otimes \dots \otimes L^{\sigma^{n-1}}.$$

Let (Z, τ) be another scheme and automorphism, and let $\phi : Z \rightarrow X$ be a morphism, such that $\sigma \circ \phi = \phi \circ \tau$. Let $N = \phi^* L$. Let

$$\tilde{B} = \bigoplus_n \Gamma(Z, N_n), \quad N_n = N \otimes N^\tau \otimes \dots \otimes N^{\tau^{n-1}}$$

be the twisted homogeneous coordinate ring of (Z, τ, N) .

We have a canonical ring morphism

$$B \rightarrow \tilde{B}.$$

In degree n it is given by the pullback map

$$(\phi^*)^{\otimes n} : \Gamma(X, L_n) \longrightarrow \Gamma(Z, N_n),$$

which exists because

$$N^{\tau^i} = (\phi^* L)^{\tau^i} = \phi^*(L^{\sigma^i}).$$

Line in X

Let $U \subset V$ be 1-dimensional, and $Z = Z(U) \subset \mathbb{P}(V)$. Assume that $Z \subset X$, and that σ factors through Z . Then we get an associated algebra quotient $B \rightarrow \tilde{B}$. The algebra \tilde{B} is a twist of the polynomial ring in two variables, or a quantum projective line.

In degree 2, we have a short exact sequence

$$0 \longrightarrow Q \longrightarrow V/U \otimes V/U \longrightarrow \tilde{B}_2 \longrightarrow 0.$$

The dimension of \tilde{B}_2 is 3, by the classification of quantum projective lines, and $\dim Q = 1$. We consider the induced morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & V \otimes V & \longrightarrow & B_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Q & \longrightarrow & V/U \otimes V/U & \longrightarrow & \tilde{B}_2 \longrightarrow 0. \end{array}$$

Let $T(V)$ be the free or tensor algebra on V , and $A = T(V)/R$ the regular algebra associated to our elliptic triple. Let $C_3 \subset A_3$ be the one-dimensional subspace generated by c_3 .

Lemma 1.12 *The map $R \rightarrow Q$ is surjective.*

PROOF. If not, it is zero. Then R is in the kernel of $V \otimes V \rightarrow V/U \otimes V/U$, which is $U \otimes V + V \otimes U$. If we choose a basis (x_i) for V , with $x_1 \in U$, and a basis (f_i) for R , then the matrix M , such that $f = Mx$ is of the form

$$M = \begin{pmatrix} m_{11} & a_{12} x_1 & a_{13} x_1 \\ m_{21} & a_{22} x_1 & a_{23} x_1 \\ m_{31} & a_{32} x_1 & a_{33} x_1 \end{pmatrix}.$$

Because X is defined by $\det M = 0$, we see that the double line $x_1^2 = 0$ is contained in X . Also, we see that along this line, the matrix M has rank at most 1. This contradicts the assumption that A is regular, by Lemma 4.4 of [3]. \square

Corollary 1.13 *We have $\tilde{B} = T(V)/\langle U, R \rangle$, where $\langle U, R \rangle$ denotes the two-sided ideal in the tensor algebra $T(V)$, generated by the subspaces $U \subset V$ and $R \subset V \otimes V$.*

PROOF. By construction, \tilde{B} comes with a morphism $T(V)/\langle U, R \rangle \rightarrow \tilde{B}$. We have to construct a morphism in the other direction.

By the classification of quantum projective lines, we know that $\tilde{B} = T(V/U)/\langle Q \rangle$. The embedding $V/U \rightarrow T(V)/\langle U \rangle$ induces an algebra morphism $T(V/U) \rightarrow T(V)/\langle U \rangle$, hence a morphism $T(V/U) \rightarrow T(V)/\langle U, R \rangle$, which, by the lemma, annihilates Q . This gives us the morphism $\tilde{B} \rightarrow T(V)/\langle U, R \rangle$, which is the required inverse. \square

Corollary 1.14 *In A_3 , we have $C_3 \subset UVV + VUV + VVU$.*

PROOF. We have the succession of quotients

$$T(V) \twoheadrightarrow A \twoheadrightarrow B \twoheadrightarrow \tilde{B},$$

which corresponds to the sequence of ideals in $T(V)$

$$0 \subset \langle R \rangle \subset \langle R, C_3 \rangle \subset \langle U, R \rangle.$$

In particular, modulo $\langle R \rangle$, we have $C_3 \subset \langle U \rangle_3 = UVV + VUV + VVU$. \square

Proof of Proposition 1.3. In B , we have that $UV = VU$. It follows that this is also true in A , as it is a claim about $A_2 = B_2$. Then it follows that $UA = AU$, because A is generated by V . From this, it follows that in fact $C_3 \subset UVV$ and $C_3 \subset VVU$. Let z be a generator of U . Then it follows that z is a left and right regular element in A , because this is true for any generator c_3 of C_3 .

The test configuration in A , generated by the flag $V \supset U \supset 0$ in A_1 , is the sequence of two-sided ideals $J^{(k)} = \langle U \rangle^k$. By what we just proved, we have $\langle U \rangle^k = z^k A \cong A(-k)$ the shift of A by $-k$.

Therefore, for the weight function $\tilde{w}(n)$ of the test configuration $(J^{(k)})$, we have

$$\tilde{w}(n) = \sum_{k=1}^n a(k-1),$$

where $a(n) = \dim A_n = \frac{1}{2}(n+1)(n+2)$. This implies that

$$\tilde{w}(n) = \frac{1}{3} n a(n),$$

or

$$\tilde{F}(n) = \frac{\tilde{w}(n)}{n a(n)} = \frac{1}{3} = \tilde{F}(1).$$

So we see that the Futaki function of (z^k) is constant. This proves Proposition 1.3. \square

Line with embedded point

Consider now \mathbb{P}^1 with a 2-dimensional embedded point. Call this scheme \tilde{Z} , and $Z = \tilde{Z}^{\text{red}} = \mathbb{P}^1$. The scheme \tilde{Z} is embedded into $\mathbb{P}(V)$ by the choice of a flag $0 \subset U \subset W \subset V$, with $\dim U = 1$ and $\dim W = 2$, as the intersection of two quadrics $\tilde{Z} = Z(UW)$. The homogeneous coordinate ring of \tilde{Z} is $\mathbb{C}[V]/(UW)$, which is, in coordinates, $\mathbb{C}[x, y, z]/(x^2, xy)$.

Lemma 1.15 *Any twisted homogeneous coordinate ring of \tilde{Z} is quadratic. More precisely, if $\sigma : \tilde{Z} \rightarrow \tilde{Z}$ is any scheme automorphism, then the associated twisted homogeneous coordinate ring is*

$$\tilde{B} = \frac{T(V)}{\langle U \rangle \langle W \rangle + \langle W \rangle \langle U \rangle + \langle Q \rangle},$$

where Q is defined by the exact sequence

$$0 \longrightarrow Q \longrightarrow \frac{V \otimes V}{U \otimes W + W \otimes U} \longrightarrow \tilde{B}_2 \longrightarrow 0.$$

PROOF. We omit the proof, which is not difficult, as this result is not used in the proof of the main theorem. \square

Assume now that $\tilde{Z} \subset X$, and that σ factors through \tilde{Z} . This will be the case, for example, if X is nodal, contains the line $Z(U) \subset X$, invariant by σ , and a node $Z(W) \subset X$ fixed by σ .

Then we get an associated algebra quotient $B \rightarrow \tilde{B}$. In degree 2, we have a short exact sequence

$$0 \longrightarrow Q \longrightarrow \frac{V \otimes V}{U \otimes W + W \otimes U} \longrightarrow \tilde{B}_2 \longrightarrow 0 .$$

The dimension of \tilde{B}_2 is 4, so $\dim Q = 2$. We consider the induced morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & V \otimes V & \longrightarrow & B_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Q & \longrightarrow & \frac{V \otimes V}{U \otimes W + W \otimes U} & \longrightarrow & \tilde{B}_2 \longrightarrow 0 . \end{array}$$

Lemma 1.16 *The map $R \rightarrow Q$ is surjective.*

PROOF. If not, the intersection of R and $UW + WU$ has dimension 2. If we choose a basis (x_i) for V , with $x_1 \in U$, and $x_2 \in W$, and a basis (f_i) for R , such that $f_1, f_2 \in UW + WU$, then the matrix M , such that $f = Mx$ is of the form

$$M = \begin{pmatrix} m_{11}(x_1, x_2) & a_{12} x_1 & 0 \\ m_{21}(x_1, x_2) & a_{22} x_1 & 0 \\ m_{31} & m_{32} & m_{33} \end{pmatrix} .$$

At the point $\langle 0, 0, 1 \rangle$, this matrix has rank 1, so this contradicts the non-degeneracy assumption, see Lemma 4.4 of [3]. \square

Corollary 1.17 *Let $T(V)$ be the free algebra on V . Then*

$$\tilde{B} = \frac{T(V)}{\langle U \rangle \langle W \rangle + \langle W \rangle \langle U \rangle + \langle R \rangle} .$$

PROOF. By the lemma, modulo $\langle W \rangle \langle U \rangle$, we have $Q = R$. \square

Corollary 1.18 *In A_3 , we have*

$$C_3 \subset UWV + UVW + VUW + WUV + VWU + WVU .$$

PROOF. We have the succession of quotients

$$T(V) \twoheadrightarrow A \twoheadrightarrow B \twoheadrightarrow \tilde{B} ,$$

which corresponds to the sequence of ideals in $T(V)$

$$0 \subset \langle R \rangle \subset \langle R, C_3 \rangle \subset \langle U \rangle \langle W \rangle + \langle W \rangle \langle U \rangle + \langle R \rangle .$$

In particular, modulo $\langle R \rangle$, we have $C_3 \subset (\langle U \rangle \langle W \rangle + \langle W \rangle \langle U \rangle)_3$. \square

Singular point

Consider now a point $P = Z(W) \in X$, fixed by σ , and assume that P is a singularity of X . As $\sigma : X \rightarrow X$ is a scheme automorphism, σ induces an automorphism of the first order neighbourhood Z of P in X , which is isomorphic to $\text{Spec } \mathbb{C}[x, y]/(x, y)^2$, as P is a singular point of X . The scheme $Z = \text{Spec } \mathbb{C}[x, y]/(x, y)^2$ is embedded into $\mathbb{P}(V)$ as $Z = Z(WW)$.

Let \tilde{B} be the twisted coordinate ring of $(Z, \sigma|_Z)$. We have quotients $A \rightarrow B \rightarrow \tilde{B}$. Choose a vector $z \in V$, $z \notin W$. Then \tilde{z} , the image of z in \tilde{B} , is a left and right regular element. Moreover, both left and right multiplication by \tilde{z} induce isomorphisms $\tilde{B}_n \rightarrow \tilde{B}_{n+1}$, for all $n \geq 2$.

Let Q be the kernel defined by the exact sequence

$$0 \longrightarrow Q \longrightarrow \frac{V \otimes V}{W \otimes W} \longrightarrow \tilde{B}_2 \longrightarrow 0 .$$

Then $\tilde{B} = T(V)/(\langle WW \rangle + \langle Q \rangle)$, as can be seen by studying the structure of \tilde{B} .

As above, regularity implies that R maps onto Q , and hence that

$$\tilde{B} = \frac{T(V)}{\langle W \rangle \langle W \rangle + \langle R \rangle} ,$$

and that

$$c_3 \in WWV + WVW + VWW . \quad (10)$$

In particular, we have that left or right multiplication by $z \in A$ induces an isomorphism

$$(A/\langle W \rangle^2)_n \xrightarrow{z} (A/\langle W \rangle^2)_{n+1} ,$$

for all $n \geq 2$.

Consider now the (doubly) graded ring

$$R = \bigoplus_{k=0}^{\infty} \langle W \rangle^k / \langle W \rangle^{k+1} .$$

In fact, R is the central fibre of the test configuration generated by W , and A is S-equivalent to R .

By the facts we proved, the subring of k -degree zero, $R^0 = A/\langle W \rangle$ is a quantum projective line, and R is generated over R^0 by one element which quantum commutes with R^0 . From this, the graded dimension of each $J^{(k)}\langle W \rangle^k$ can be computed, giving a proof of Proposition 1.4.

1.5 More analysis of c_3

Let (X, σ, L) be a regular elliptic triple, such that $L \not\cong L^\sigma$.

Recall the notion of **tame** line bundle, defined in [3]. A line bundle on X is tame if either H^0 vanishes, or H^1 vanishes, or if it is trivial. For tame line bundles, h^0 and h^1 can be calculated as \deg or $-\deg$, respectively. Bundles

generated by sections, duals of tame bundles, and bundles with non-negative degree on each component are all tame. For example, every non-trivial tame line bundle of degree 0 has $H^0 = H^1 = 0$.

As in [3], for a sheaf F , generated by global sections, we denote by F'' the kernel of the epimorphism $\Gamma(X, F) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow F$. The reason for introducing F'' is the following Proposition 7.17 of [3].

Proposition 1.19 *If M is a coherent \mathcal{O}_X -module generated by global sections, and N is locally free with $H^1(X, N) = 0$, then the kernel and cokernel of the multiplication map*

$$\Gamma(X, M) \otimes_k \Gamma(X, N) \longrightarrow \Gamma(X, M \otimes_{\mathcal{O}_X} N)$$

are given by $H^0(X, M'' \otimes_{\mathcal{O}_X} N)$ and $H^1(X, M'' \otimes_{\mathcal{O}_X} N)$, respectively.

If we apply this to $M = L$ and $N = L^\sigma$, we get the exact sequence

$$0 \longrightarrow \Gamma(L'' \otimes L^\sigma) \longrightarrow \Gamma(L) \otimes \Gamma(L^\sigma) \longrightarrow \Gamma(L \otimes L^\sigma) \longrightarrow 0 .$$

If we identify A_2 with $\Gamma(L \otimes L^\sigma)$, this identifies $\Gamma(L'' \otimes L^\sigma)$ with R , the kernel of multiplication $V \otimes V \rightarrow A_2$. By the regularity of A , we can write A_3 as the cokernel of $R \otimes A_1 \rightarrow V \otimes A_2$. This gives the exactness of the lower row in the following diagram:

$$\begin{array}{ccccccc} \Gamma(L'' \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) & \xrightarrow{\beta} & \Gamma(L'' \otimes L^\sigma \otimes L^{\sigma^2}) & \longrightarrow & C_3 & \longrightarrow & 0 \\ \downarrow = & & \downarrow & & \downarrow & & \\ \Gamma(L'' \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) & \longrightarrow & \Gamma(L) \otimes \Gamma(L^\sigma \otimes L^{\sigma^2}) & \longrightarrow & A_3 & \longrightarrow & 0 . \end{array} \quad (11)$$

The exactness of the upper row comes from Lemma 7.29 of [3].

The case where P lies on X

Consider a 2-dimensional subspace $W \subset \Gamma(L)$, which does *not* generate L . Let $P \in X$ be the point where W fails to generate L . (Mapping P into \mathbb{P}^2 , via the embedding $X \rightarrow \mathbb{P}^2$, we obtain the point in \mathbb{P}^2 , dual to the hyperplane W in V .) We have a surjection of sheaves on X

$$W \otimes_{\mathbb{C}} \mathcal{O}_X \twoheadrightarrow L \otimes_{\mathcal{O}_X} \mathfrak{m}_P = \mathfrak{m}_P L . \quad (12)$$

If P is a smooth point of X , then P is a Cartier divisor, and hence $\mathfrak{m}_P L = L(-P)$, but we do not want to make this assumption.

We note also that $\Gamma(L \otimes_{\mathcal{O}_X} \mathfrak{m}_P) = W$, and that $\mathfrak{m}_P L$ is generated by global sections. We consider $(\mathfrak{m}_P L)''$:

$$0 \longrightarrow (\mathfrak{m}_P L)'' \longrightarrow W \otimes \mathcal{O}_X \longrightarrow \mathfrak{m}_P L \longrightarrow 0$$

If P is a smooth point of X , then $(\mathfrak{m}_P L)''$ is locally free of rank 1, but not otherwise.

Lemma 1.20 *Any non-zero element of $\Lambda^2 W$ induces an isomorphism*

$$(\mathfrak{m}_P L)'' \xrightarrow{\sim} (\mathfrak{m}_P L)^\vee .$$

PROOF. Let us choose a non-zero element in $\Lambda^2 W$. As $\Lambda^2 W$ is one-dimensional, this element is a basis element, and also induced a basis element in the dual space $\Lambda^2 W^\vee$. Let us denote this element by κ , it is a non-degenerate alternating pairing on W , and induces a non-degenerate alternating pairing on the trivial bundle $W \otimes \mathcal{O}_X$.

We claim that $(\mathfrak{m}_P L)'' \subset W \otimes \mathcal{O}_X$ is isotropic with respect to κ . As κ takes values in \mathcal{O}_X , and \mathfrak{m}_P contains a non-zero divisor of \mathcal{O}_X , we can check this claim after removing P from X , where it becomes trivial, as all sheaves involved become locally free. Thus, κ induces a pairing

$$\kappa : (\mathfrak{m}_P L)'' \otimes_{\mathcal{O}_X} \mathfrak{m}_P L \longrightarrow \mathcal{O}_X .$$

This pairing is non-degenerate, i.e., induces injections $(\mathfrak{m}_P L)'' \rightarrow (\mathfrak{m}_P L)^\vee$ and $\mathfrak{m}_P L \rightarrow (\mathfrak{m}_P L)''^\vee$. To prove these facts, it is again enough to restrict to $X - \{P\}$, because both $(\mathfrak{m}_P L)''$ and $\mathfrak{m}_P L$ are submodules of locally free finite rank \mathcal{O}_X -modules.

By the snake lemma, the cokernel of $(\mathfrak{m}_P L)'' \rightarrow (\mathfrak{m}_P L)^\vee$ is equal to the kernel of $\mathfrak{m}_P L \rightarrow (\mathfrak{m}_P L)''^\vee$, and hence vanishes. \square

Lemma 1.21 *We have*

$$H^1((\mathfrak{m}_P L)'' \otimes L^\sigma) = 0, \quad \text{and} \quad H^1((\mathfrak{m}_P L)'' \otimes L^\sigma \otimes L^{\sigma^2}) = 0 .$$

PROOF. By the previous lemma, there exists a monomorphism $L^{-1} \rightarrow (\mathfrak{m}_P L)^\vee \cong (\mathfrak{m}_P L)''$, whose cokernel is supported over $\{P\}$. Hence we get induced epimorphisms $H^1(L^{-1} \otimes L^\sigma) \rightarrow H^1((\mathfrak{m}_P L)'' \otimes L^\sigma)$ and $H^1(L^{-1} \otimes L^\sigma \otimes L^{\sigma^2}) \rightarrow H^1((\mathfrak{m}_P L)'' \otimes L^\sigma \otimes L^{\sigma^2})$. Now it suffices to remark that by Lemma 7.18 of [3], both line bundles $L^{-1} \otimes L^\sigma$ and $L^{-1} \otimes L^\sigma \otimes L^{\sigma^2}$ are tame, and so their respective H^1 vanishes; in the first case because $L^{-1} \otimes L^\sigma$ is a non-trivial line bundle of degree 0, and in the second case because $L^{-1} \otimes L^\sigma \otimes L^{\sigma^2}$ is of degree 3. \square

Lemma 1.22 *The sheaf $\mathfrak{m}_P L^\sigma$ is also generated by its global sections. We have $(\mathfrak{m}_P L^\sigma)'' \cong (\mathfrak{m}_P L^\sigma)^\vee$, as well as $H^1((\mathfrak{m}_P L^\sigma)'' \otimes L^{\sigma^2}) = 0$.*

PROOF. The same proofs apply. \square

Choosing a basis for V/W , we get the exact square

$$\begin{array}{ccccc}
 (\mathfrak{m}_P L)'' & \longrightarrow & W \otimes \mathcal{O}_X & \longrightarrow & \mathfrak{m}_P L \\
 \downarrow & & \downarrow & & \downarrow \\
 L'' & \longrightarrow & \Gamma(L) \otimes \mathcal{O}_X & \longrightarrow & L \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathfrak{m}_P & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_P
 \end{array} \tag{13}$$

For $N = L^\sigma$ and $N = L^\sigma \otimes L^{\sigma^2}$, we consider the induced short exact sequence

$$0 \longrightarrow (\mathfrak{m}_P L)'' \otimes N \longrightarrow L'' \otimes N \longrightarrow \mathfrak{m}_P N \longrightarrow 0 .$$

In both cases, we get an induced short exact sequence of vector spaces

$$0 \longrightarrow \Gamma((\mathfrak{m}_P L)'' \otimes N) \longrightarrow \Gamma(L'' \otimes N) \longrightarrow \Gamma(\mathfrak{m}_P N) \longrightarrow 0 ,$$

because $H^1((\mathfrak{m}_P L)'' \otimes N) = 0$, by Lemma 1.21.

We obtain the morphism of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Gamma((\mathfrak{m}_P L)'' \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) & \longrightarrow & \Gamma(L'' \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) & \longrightarrow & \\
 & & \downarrow \alpha & & \downarrow \beta & & \\
 0 & \longrightarrow & \Gamma((\mathfrak{m}_P L)'' \otimes L^\sigma \otimes L^{\sigma^2}) & \longrightarrow & \Gamma(L'' \otimes L^\sigma \otimes L^{\sigma^2}) & \longrightarrow & \\
 & & & & \longrightarrow & \Gamma(\mathfrak{m}_P L^\sigma) \otimes \Gamma(L^{\sigma^2}) & \longrightarrow 0 \\
 & & & & & \downarrow \gamma & \\
 & & & & & \longrightarrow & \Gamma(\mathfrak{m}_P L^\sigma \otimes L^{\sigma^2}) \longrightarrow 0 .
 \end{array}$$

By Lemma 1.22, γ is surjective. It follows that $\text{cok } \alpha \rightarrow \text{cok } \beta$ is an epimorphism. Hence we get a morphism of exact sequences

$$\begin{array}{ccccccc}
 \Gamma((\mathfrak{m}_P L)'' \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) & \xrightarrow{\alpha} & \Gamma((\mathfrak{m}_P L)'' \otimes L^\sigma \otimes L^{\sigma^2}) & \longrightarrow & \text{cok } \alpha & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \Gamma(L'' \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) & \xrightarrow{\beta} & \Gamma(L'' \otimes L^\sigma \otimes L^{\sigma^2}) & \longrightarrow & C_3 & \longrightarrow & 0
 \end{array} \tag{14}$$

Composing (14) with (11), we obtain the morphism of exact sequences

$$\begin{array}{ccccccc}
\Gamma((\mathfrak{m}_P L)'' \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) & \longrightarrow & \Gamma((\mathfrak{m}_P L)'' \otimes L^\sigma \otimes L^{\sigma^2}) & \longrightarrow & \text{cok } \alpha & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Gamma(L'' \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) & \longrightarrow & \Gamma(L) \otimes \Gamma(L^\sigma \otimes L^{\sigma^2}) & \longrightarrow & A_3 & \longrightarrow & 0
\end{array} \tag{15}$$

By construction, $\Gamma((\mathfrak{m}_P L)'' \otimes L^\sigma \otimes L^{\sigma^2}) \rightarrow \Gamma(L) \otimes \Gamma(L^\sigma \otimes L^{\sigma^2})$ factors through $W \otimes \Gamma(L^\sigma \otimes L^{\sigma^2})$. We conclude:

Proposition 1.23 *If $P = Z(W) \in X$, then in A_3 , we have $c_3 \in WV$.*

Smooth point

To go further, let us now assume that P is a smooth point of X . Then we have $\mathfrak{m}_P L = L(-P)$, and $(\mathfrak{m}_P L)'' = L^{-1}(P)$.

Consider the lattice of subspaces

$$\begin{array}{ccc}
& \Gamma(L) \otimes \Gamma(L^\sigma \otimes L^{\sigma^2}) & \\
& \uparrow & \\
& W \otimes \Gamma(L^\sigma \otimes L^{\sigma^2}) & \\
& \uparrow & \swarrow \\
\Gamma(L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2}) & & W \otimes \Gamma(L^\sigma \otimes L^{\sigma^2}(-T)) \\
\uparrow \alpha & \swarrow & \uparrow \\
\Gamma(L^{-1}(P) \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) & & \Gamma(L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2}(-T)).
\end{array}$$

Here T is an effective Cartier divisor of degree 1 or 2.

Lemma 1.24 *Let $Q \in X^{\text{reg}}$ be the unique point such that $L^\sigma(P) = L(Q)$. Suppose that $Q \notin \text{supp } T$, and that $L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2}(-T)$ is generated by global sections. Then we have*

$$\begin{aligned}
& \Gamma(L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2}) \\
&= \Gamma(L^{-1}(P) \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) + \Gamma(L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2}(-T)).
\end{aligned}$$

PROOF. The line bundle $L^{-1}(P) \otimes L^\sigma$ has degree 1, so it has an essentially unique non-zero section, which we shall call $g \in \Gamma(L^{-1}(P) \otimes L^\sigma)$. It vanishes exactly at the point Q . We conclude that $\Gamma(L^{-1}(P) \otimes L^\sigma) = \Gamma(L^{-1}(P) \otimes L^\sigma(-Q))$. Therefore the image of $\Gamma(L^{-1}(P) \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2})$ in $\Gamma(L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2})$ is contained in $\Gamma(L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2}(-Q))$.

As the bundle $L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2}(-T)$ is generated by global sections, $\Gamma(L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2}(-T))$ is not contained in $\Gamma(L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2}(-Q))$. This implies the lemma for dimension reasons. \square

Corollary 1.25 *Under the same assumptions, as subspaces of $W \otimes \Gamma(L^\sigma \otimes L^{\sigma^2})$, we have*

$$\Gamma(L^{-1}(P) \otimes L^\sigma \otimes L^{\sigma^2}) \subset \Gamma(L^{-1}(P) \otimes L^\sigma) \otimes \Gamma(L^{\sigma^2}) + W \otimes \Gamma(L^\sigma \otimes L^{\sigma^2}(-T)).$$

Corollary 1.26 *If $P = Z(W)$ is a smooth point of X such that $\sigma(P) = P$, then in A_3 , we have $c_3 \in WVW$.*

PROOF. This follows from the above considerations, upon taking $T = P$, in particular Sequence (14), and the fact that $V \otimes W \rightarrow \Gamma(L^\sigma \otimes L^{\sigma^2}(-P))$ is surjective. \square

Corollary 1.27 *If $P = Z(W)$ is a smooth point of X , and σ is a translation, of order not equal to 2, then in A_3 , we have $c_3 \in WWW$.*

PROOF. If σ is the translation by the point $S \in X^{\text{reg}}$, then $Q = P - 3S$, in the group law on X^{reg} . Moreover, $P^\sigma = P - S$, and $P^{\sigma^2} = P - 2S$, so the lemma and its corollary apply to $T = P + P^\sigma$, as $2S \neq 0$.

We also use that fact that $W \otimes W \rightarrow \Gamma(L^\sigma(-P^\sigma) \otimes L^{\sigma^2}(-P^{\sigma^2}))$ is surjective, which follows from Lemma 2.1, below. \square

1.6 Non-special points

This is the case where if $P \in X$, then $\sigma(P) \neq P$. We do not exclude the case that P is a singular point of X .

Lemma 1.28 *Let W be a non-special 2-dimensional subspace of $V = B_1$. In B_2 we have $WV + VW = VV = B_2$.*

Moreover, if $P \notin X$, we have the stronger result that $B_2 = WV$.

PROOF. If the base locus of W is empty, W generates L , and we have a short exact sequence of vector bundles on X :

$$0 \longrightarrow M \longrightarrow W \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0.$$

Here M is a line bundle which is isomorphic to L^{-1} , but not canonically so. Tensoring with L^σ and taking global sections gives us the exact sequence

$$W \otimes \Gamma(L^\sigma) \longrightarrow \Gamma(L \otimes L^\sigma) \longrightarrow H^1(M \otimes L^\sigma).$$

As $L \not\cong L^\sigma$, we have $M \otimes L^\sigma \not\cong \mathcal{O}_X$, and hence $H^1(M \otimes L^\sigma) = 0$, and so $W \otimes \Gamma(L^\sigma) \rightarrow \Gamma(L \otimes L^\sigma)$ is surjective. This proves that $WV = B_2$. (In

degenerate cases, this proof uses the fact that $L^{-1} \otimes L^\sigma$ is *tame*, in the language of [3], which is proved in Lemma 7.18 of [ibid.].)

For a line bundle M on X , and reduced point $Q \in X$, the set of sections of M vanishing at Q is $\Gamma(X, M \otimes \mathfrak{m}_Q) \subset \Gamma(X, M)$. (We do not use the notation $\Gamma(X, M(-Q))$ to include the case where Q is a singularity, and hence not a Cartier divisor.) If M has sections which do not vanish at Q , then $\dim \Gamma(M \otimes \mathfrak{m}_Q) = \dim \Gamma(M) - 1$.

Now suppose that W has a base point $P \in X$. The base locus of $W' = \sigma^*W \subset \Gamma(X, \sigma^*L)$ is $P' = \sigma^{-1}P$. We have $W = \Gamma(L \otimes \mathfrak{m}_P) \subset \Gamma(L)$ and $W' = \Gamma(L^\sigma \otimes \mathfrak{m}_{P'}) \subset \Gamma(L^\sigma)$. Because $P' \neq P$, the line bundle $L \otimes L^\sigma$ has sections which vanish at P but not at P' , and sections which vanish at P' but not at P . Thus $\Gamma(L \otimes L^\sigma \otimes \mathfrak{m}_P)$ and $\Gamma(L \otimes L^\sigma \otimes \mathfrak{m}_{P'})$ are two distinct codimension 1 subspaces of $\Gamma(L \otimes L^\sigma)$. Their sum is then necessarily the whole space:

$$\Gamma(L \otimes \mathfrak{m}_P \otimes L^\sigma) + \Gamma(L \otimes L^\sigma \otimes \mathfrak{m}_{P'}) = \Gamma(L \otimes L^\sigma). \quad (16)$$

We now remark that the image of $\Gamma(L \otimes \mathfrak{m}_P) \otimes \Gamma(L^\sigma)$ in $\Gamma(L \otimes L^\sigma)$ is equal to $\Gamma(L \otimes \mathfrak{m}_P \otimes L^\sigma)$. (This follows formally from the fact that $\Gamma(L) \otimes \Gamma(L^\sigma) \rightarrow \Gamma(L \otimes L^\sigma)$ is surjective, and that $\Gamma(L \otimes \mathfrak{m}_P) \otimes \Gamma(L^\sigma) \subset \Gamma(L) \otimes \Gamma(L^\sigma)$ and $\Gamma(L \otimes L^\sigma \otimes \mathfrak{m}_P) \subset \Gamma(L \otimes L^\sigma)$ are both codimension 1 subspaces.) Thus a reformulation of (16) is $WV' + VW' = VV'$. This is our claim. \square

An application to the study of c_3

Theorem 1.29 *Let $W \subset V$ be an arbitrary 2-dimensional subspace. In A_3 , we have $c_3 \in WWV + WVW + VWV$.*

PROOF. Let $P = Z(W)$. If $P \notin X$, we have $VV = WV$, and hence $A_3 = VVV = WVW = WWV$, and the claim is trivial. If $P \in X$, but $\sigma(P) \neq P$, we use Proposition 1.23, to deduce $c_3 \in WVW = W(WV + VW)$. This leaves the case where $P \in X$, and $\sigma(P) = P$. If P is a non-singular point, our claim follows from Corollary 1.26. If P is a singular point, it follows from (10). \square

We also need the following amplification:

Proposition 1.30 *Suppose X is a Neron triangle, and σ permutes the sides of X cyclically. Assume that $P = Z(W)$ is a point on X . Then in A_3 we have $c_3 \in WWW$.*

PROOF. Fix a base point for X , making X^{reg} a group scheme, and σ a translation. Then the order of σ is not equal to 2. So if P is a smooth point, the claim follows from Corollary 1.27. If P is a node, we can deduce the proof from the case where P is smooth by a degeneration argument. \square

2 Estimates

Here we prove the stability estimates claimed in Proposition 1.5, by finding suitable values for the functions $w_\ell(n)$ and $w_m(n)$, and then proving Formula (8).

We may assume that X is either smooth, or a Neron triangle, on which σ acts by cyclically permuting the sides.

We always use notation $P = Z(W)$ and $Y = Z(U)$, and treat the following cases in turn:

- (i) P lies off X ,
- (ii) P lies on X , but is neither a node of X nor a fixed point of σ ,
- (iii) $P \in X^{\text{reg}}$ is a fixed point of σ ,
- (iv) P is a node of X ,
- (v) Y is a linear component of X .

In the cases (i) to (iv), we assume $Y \not\subset X$. Most of these cases are divided into two subcases, a case where ℓ is large compared to m , which includes the case $m = 0$, and the converse case.

2.1 The main dimension estimate

Lemma 2.1 *Let L and M be line bundles on X , both generated by global sections, and both of degree at least 2. The multiplication map*

$$\Gamma(L) \otimes \Gamma(M) \longrightarrow \Gamma(L \otimes M)$$

is surjective, unless $\deg L = \deg M = 2$ and $L \cong M$.

PROOF. Because of Proposition 1.19, an equivalent statement is that

$$H^1(L'' \otimes M) = 0.$$

The proof will proceed by induction on the integer $\deg L + \deg M$. The base case is when $\deg L = \deg M = 2$. In this case, $L'' \cong L^{-1}$ is a tame line bundle. Then $L'' \otimes M$ is also tame. Hence $H^1(L'' \otimes M)$ vanishes if and only if $L \not\cong M$.

Now suppose that $\deg L > 2$. Find a point $P \in X$, such that $L(-P)$ is still generated by global sections, and such that $L(-P) \not\cong M$. We can apply the induction hypothesis to the bundles $L(-P)$ and M , obtaining that

$$H^1(L(-P) \otimes M'') = 0.$$

But there is a surjection

$$H^1(L(-P) \otimes M'') \twoheadrightarrow H^1(L \otimes M'')$$

proving that $H^1(L \otimes M'') = 0$. \square

Corollary 2.2 *Let L_1, \dots, L_n be line bundles of degree at least 2 on X , all generated by global sections. The map*

$$\Gamma(L_1) \otimes \dots \otimes \Gamma(L_n) \longrightarrow \Gamma(L_1 \otimes \dots \otimes L_n)$$

is surjective, unless all L_i are of degree 2 and pairwise isomorphic to each other.

PROOF. Assume that the bundles are not all of degree 2 or not all isomorphic to one another. Then, after relabelling, we may assume that if $\deg L_1 = \deg L_2 = 2$, then $L_1 \not\cong L_2$. Then we can apply the lemma successively to $L_1 \otimes \dots \otimes L_i$ and L_{i+1} , for $i = 1, \dots, n-1$. \square

2.2 The test configuration of B generated by a filtration

We will write

$$\{V^\gamma W^\alpha U^\beta\} \subset B_{\gamma+\alpha+\beta}$$

do denote the sum over all products $W_1 \dots W_{\gamma+\alpha+\beta}$, where γ of the W_i are equal to V , and α of the W_i are equal to W , and β of the W_i are equal to U . For example,

$$\{VWU\} = VWU + VUW + WUV + WVU + UVW + UWV \subset B_3.$$

Lemma 2.3 *Suppose that $\ell > 0$ and $m > 0$. Then we have*

$$I_n^{(k)} = \sum_{\alpha, \beta} \{V^{n-\alpha-\beta} W^\alpha U^\beta\},$$

where (α, β) ranges over all pairs of non-negative integers satisfying

- (i) $\alpha + \beta \leq n$,
- (ii) $\alpha + \beta(\ell + 1) \leq k \leq \alpha\ell + \beta(\ell + m)$.

PROOF. By construction, $\{V^{n-\alpha-\beta} W^\alpha U^\beta\}$ partakes in $I_n^{(k)}$ if and only if there exist integers $0 < i_1, \dots, i_\alpha \leq \ell$ and $\ell < j_1, \dots, j_\beta \leq \ell + m$ such that $k = i_1 + \dots + i_\alpha + j_1 + \dots + j_\beta$. This immediately proves the stated bounds on k .

Conversely, if (ii) is satisfied, we can write $k = k_1 + k_2$, with $\alpha \leq k_1 \leq \alpha\ell$ and $\beta(\ell + 1) \leq k_2 \leq \beta(\ell + m)$. Then we can write $k_1 = i_1 + \dots + i_\alpha$ with $0 < i_1, \dots, i_\alpha \leq \ell$ and $k_2 = j_1 + \dots + j_\beta$ with $\ell < j_1, \dots, j_\beta \leq \ell + m$. \square

Lemma 2.4 *Suppose that $\ell > 0$ and $m = 0$. Then we have*

$$I_n^{(k)} = \sum_{\alpha} \{V^{n-\alpha} W^\alpha\},$$

where α ranges over all integers such that $\alpha \leq n$ and $\alpha \leq k \leq \alpha\ell$. \square

Lemma 2.5 *Suppose that $\ell = 0$ and $m > 0$. Then we have*

$$I_n^{(k)} = \sum_{\beta} \{V^{n-\beta} U^\beta\},$$

where β ranges over all integers satisfying $\beta \leq n$ and $\beta \leq k \leq \beta m$. \square

Useful reformulations: the W -case

Lemma 2.6 *For all $\alpha = 1, \dots, n$ we have*

- (i) *if $(\alpha - 1)\ell < k \leq \alpha\ell$ then $\{V^{n-\alpha}W^\alpha\} \subset I_n^{(k)}$,*
- (ii) *if $n\ell + (\alpha - 1)m < k \leq n\ell + \alpha m$ then $\{W^{n-\alpha}U^\alpha\} \subset I_n^{(k)}$.*

PROOF. The hardest claim is that $n\ell + (\alpha - 1)m < k$ implies $(n - i) + \alpha(\ell + 1) \leq k$. This is proved by noting that under our assumptions

$$0 \leq (n - \alpha)(\ell - 1) + (m - 1)(\alpha - 1).$$

Reformulating gives

$$(n - \alpha) + \alpha(\ell + 1) \leq n\ell + (\alpha - 1)m + 1,$$

which is what we need. \square

The following variation will also be needed:

Lemma 2.7 *For all $i = 1, \dots, n - 1$, and every k , such that*

$$(i - 1)\ell < k \leq i\ell,$$

we have $\{V^{n-i}W^i\} \subset I_n^{(k)}$.

For all $i = 1, \dots, n - 1$, and all k which satisfy

$$(n - 1)\ell + (i - 1)m < k \leq (n - 1)\ell + im \tag{17}$$

we have $\{VW^{n-1-i}U^i\} \subset I_n^{(k)}$.

For $(n - 1)(\ell + m) < k \leq (n - 1)(\ell + m) + \ell$, we have $\{WU^{n-1}\} \subset I_n^{(k)}$, and for $(n - 1)(\ell + m) + \ell < k \leq n(\ell + m)$, we have $U^n \subset I_n^{(k)}$.

PROOF. Let us check the second claim. We have that

$$0 \leq (\ell - 1)(n - i - 1) + (m - 1)(i - 1),$$

because each of the four quantities involved is individually non-negative, by our assumptions. Rewriting this inequality gives

$$(n - 1 - i) + i(\ell + 1) \leq (n - 1)\ell + (i - 1)m + 1. \tag{18}$$

We also have

$$(n - 1)\ell + im \leq (n - 1 - i)\ell + i(\ell + m). \tag{19}$$

Now suppose that k satisfies (17). Then, in fact,

$$(n - 1)\ell + (i - 1)m + 1 \leq k \leq (n - 1)\ell + im.$$

From (18) and (19) we conclude

$$(n - 1 - i) + i(\ell + 1) \leq k \leq (n - 1 - i)\ell + i(\ell + m).$$

With $\alpha = n - 1 - i$ and $\beta = i$ this is

$$\alpha + \beta(\ell + 1) \leq k \leq \alpha\ell + \beta(\ell + m),$$

which is the condition we need. \square

The U -case

Lemma 2.8 *Suppose that $m > 0$. For all $i = 1, \dots, n$ we have*

- (i) *if $(i-1)(\ell+m) < k \leq i(\ell+m) - m$ then $\{V^{n-i}WU^{i-1}\} \subset I_n^{(k)}$,*
- (ii) *if $i(\ell+m) - m < k \leq i(\ell+m)$ then $\{V^{n-i}U^i\} \subset I_n^{(k)}$.*

PROOF. Similar. \square

2.3 Exploiting non-commutativity

Let $Z(U) \not\subset X$. The main dimension estimate Corollary 2.2, gives that

$$\begin{aligned} \dim V^{n-\alpha-\beta}W^\alpha U^\beta &\geq 3(n-\alpha-\beta) + 2\alpha \\ &= 3n - \alpha - 3\beta, \end{aligned}$$

at least if $n - \alpha - \beta \geq 1$. We can improve this estimate for $\{V^{n-\alpha-\beta}W^\alpha U^\beta\}$.

Proposition 2.9 *We have*

$$(i) \text{ for } 0 \leq \beta \leq \frac{n}{2}, \quad \dim\{V^{n-\beta}U^\beta\} \geq 3n - 2\beta,$$

$$(ii) \text{ for } \frac{n}{2} \leq \beta \leq n, \quad \dim\{V^{n-\beta}U^\beta\} \geq 4n - 4\beta.$$

$$(iii) \text{ for } 1 \leq \beta \leq \frac{n+1}{2}, \quad \dim\{V^{n-\beta}WU^{\beta-1}\} \geq 3n + 1 - 2\beta,$$

$$(iv) \text{ for } \frac{n+1}{2} \leq \beta \leq n, \quad \dim\{V^{n-\beta}WU^{\beta-1}\} \geq 4n + 2 - 4\beta.$$

If $P = Z(W)$ is a smooth point of X , such that $\sigma(P) = P$, we also have

$$(i) \text{ for } 0 \leq \beta \leq \frac{n}{2} \quad \dim\{W^{n-\beta}U^\beta\} \geq 2n - \beta,$$

$$(ii) \text{ for } \frac{n}{2} \leq \beta \leq n \quad \dim\{W^{n-\beta}U^\beta\} \geq 3n - 3\beta.$$

PROOF. Choosing a non-zero element $s \in U$ defines an injection $\mathcal{O}_X \rightarrow L$, which identifies L with $\mathcal{O}(D)$, where D is the Cartier divisor on X , defined by the vanishing of s . Let E be the greatest common divisor of D and D^σ . Because $D \neq D^\sigma$, as $L \not\cong L^\sigma$, the degree of E is at most 2. Let

$$H = \Gamma(X, L \otimes L^\sigma(-E)) \subset \Gamma(L \otimes L^\sigma).$$

Then an argument counting dimensions, using that $UV = \Gamma(L \otimes L^\sigma(-D))$ and $VU = \Gamma(L \otimes L^\sigma(-D^\sigma))$, proves $\dim H = 6 - \deg E \geq 4$, and that that inside B_2 we have

$$H = UV + VU.$$

Now for $0 \leq \beta \leq \frac{n}{2}$, we have

$$\{V^{n-\beta}U^\beta\} \supset \{V^\beta U^\beta\}V^{n-2\beta} \supset H^\beta V^{n-2\beta},$$

and hence

$$\begin{aligned} \dim\{V^{n-\beta}U^\beta\} &\geq \dim H^\beta V^{n-2\beta} \\ &\geq 3(n-2\beta) + \beta \dim H \\ &\geq 3(n-2\beta) + 4\beta \\ &= 3n - 2\beta, \end{aligned}$$

using Lemma 2.2.

Similarly, for $\frac{n}{2} \leq \beta \leq n$, we have

$$\{V^{n-\beta}U^\beta\} \supset \{V^{n-\beta}U^{n-\beta}\}U^{2\beta-n} \supset H^{n-\beta}U^{2\beta-n},$$

and hence

$$\begin{aligned} \dim\{V^{n-\beta}U^\beta\} &\geq \dim H^{n-\beta}U^{2\beta-n} \\ &\geq 4(n-\beta) \\ &\geq 4n - 4\beta. \end{aligned}$$

For the next two claims, involving a factor of W , we proceed similarly. Let us explicate the case $1 \leq \beta \leq \frac{n+1}{2}$. Here we have

$$\begin{aligned} \{V^{n-\beta}WU^{\beta-1}\} &\supset \{V^{n-\beta}U^{\beta-1}\}W \supset \\ &\{V^{\beta-1}U^{\beta-1}\}V^{n+1-2\beta}W \supset H^{\beta-1}V^{n+1-2\beta}W, \end{aligned}$$

and hence

$$\begin{aligned} \dim\{V^{n-\beta}WU^{\beta-1}\} &\geq \dim H^{\beta-1}V^{n+1-2\beta}W \\ &\geq 3(n+1-2\beta) + (\beta-1) \dim H + 2 \\ &\geq 3n + 5 - 6\beta + 4(\beta-1) \\ &= 3n + 1 - 2\beta, \end{aligned}$$

using Lemma 2.2, or in the case that W is a node, Lemma 2.10, below.

The last two claims have the same proof as the first two, except for we replace L by $L(-P)$, throughout. This means we replace D by $D - P$, and the greatest common divisor E of $D - P$ and $D^\sigma - P$ is of degree at most 1. The line bundle $L(-1) \otimes L^\sigma(-P)(-E)$ has degree at least 3, and therefore the proof goes through, mutatis mutandis. \square

2.4 Case: P off X

This is the case where $P = Z(W)$ is not on X . Then $Y = Z(U)$ cannot be a component of X , so it intersects X in a Cartier divisor.

Let us first deal with the case where $m = 0$. By Lemma 2.6, for every $\alpha = 1, \dots, n$, we have ℓ instances of k where $\{V^{n-\alpha}W^\alpha\} \subset I_n^{(k)}$. For $\alpha < n$, we use Lemma 1.28 to conclude that $V^{n-\alpha}W^\alpha = V^n = B_n$, which has dimension $3n$. For $\alpha = n$, the estimate $\dim W^n \geq 2$, will be sufficient. Therefore, we may use

$$\begin{aligned} w_\ell(n) &= 3n(n-1) + 2 \\ &= 3n^2 - 3n + 2. \end{aligned} \tag{20}$$

Now by Theorem 1.29, we have that $c_3 \in J_3^{(2\ell)}$, and so we compute

$$\begin{aligned} (w_\ell + 2\tilde{a})(n) &= 3n^2 - 3n + 2 + (n-1)(n-2) \\ &= 4n^2 - 6n + 4, \end{aligned}$$

and then

$$\begin{aligned} G_\ell(n) &= 4\zeta_2(n) - 6\zeta_1(n) + 4\zeta_0(n) - n(n+1)(n+2) \\ &= 4\zeta_2(n) - 6\zeta_1(n) + 4\zeta_0(n) - 3\zeta_2(n) + 3\zeta_1(n) - 2\zeta_0(n) \\ &= \zeta_2(n) - 3\zeta_1(n) + 2\zeta_0(n). \end{aligned}$$

This is 0, for $n = 2$, and positive for $n \geq 3$, which is what we needed to prove.

Now if $m > 0$, we use Lemma 2.7. Then for every $\alpha = 1, \dots, n-1$, we have ℓ instances of k where $\{V^{n-\alpha}W^\alpha\} \subset I_n^{(k)}$. We also have ℓ instances of k where $WU^{n-1} \subset I_n^{(k)}$, which has dimension 2. Thus, we get the same estimate (20) we used above, leading to the same value for G_ℓ :

$$G_\ell(n) = \zeta_2(n) - 3\zeta_1(n) + 2\zeta_0(n).$$

We also get that for every $\beta = 1, \dots, n-1$, there are m instances of k where $\{VW^{n-1-\beta}U^\beta\} \subset I_n^{(k)}$. Again by Lemma 1.28, we have $VW^{n-1-\beta}U^\beta = V^{n-\beta}U^\beta$, which has dimension $3(n-\beta)$, by Lemma 2.2. We add another m instances of U^n , which has dimension 1. This gives us

$$\begin{aligned} w_m(n) &= \sum_{\beta=1}^{n-1} (3n - 3\beta) + 1 \\ &= \frac{3}{2}n^2 - \frac{3}{2}n + 1, \end{aligned}$$

and

$$\begin{aligned} G_m(n) &= \frac{3}{2}\zeta_2(n) - \frac{3}{2}\zeta_1(n) + \zeta_0(n) - \frac{1}{2}n(n+1)(n+2) \\ &= 0. \end{aligned}$$

Together with the above, we get

$$\begin{aligned} \frac{1}{\ell}(\ell G_\ell(n) + m G_m(n)) &= G_\ell(n) + \frac{m}{\ell} G_m(n) \\ &= \zeta_2(n) - 3\zeta_1(n) + 2\zeta_0(n), \end{aligned}$$

which we have already remarked is 0, for $n = 2$, and positive for $n \geq 3$. This finishes the case where P is not on X .

2.5 Case: P not a node, not a fixed point

Now we assume that $P = Z(W)$ is on X , but is neither a singularity of X , nor a fixed point of σ . We also assume that $Y = Z(U)$ is not contained in X , so it defines a Cartier divisor on X .

We distinguish to subcases: $\ell \geq m \geq 0$, and $m > \ell \geq 0$.

In the subcase $\ell \geq m \geq 0$, the main fact we use about W is that $VV = VW + WV$ in $B_2 = A_2$, see Lemma 1.28. We also use the main estimate Lemma 2.2.

For the subcase $m > \ell \geq 0$, we exploit the non-commutativity for $\{V^{n-\beta}U^\beta\}$ and $\{V^{n-\beta}WU^{\beta-1}\}$, using Lemma 2.9.

Finally, it will be important that $c_3 \in I_3^{(2\ell)}$.

Subcase $\ell \geq m \geq 0$

Let us first deal with the case where $m = 0$. Then by Lemma 2.6, for each $\alpha = 1, \dots, n$, we have ℓ instances of k , such that $\{V^{n-\alpha}W^\alpha\} \subset I_n^{(k)}$. By Lemma 1.28, we have

- (i) for $1 \leq \alpha \leq \lfloor \frac{n}{2} \rfloor$, that $\{V^{n-\alpha}W^\alpha\} = V^n = B_n$, which has dimension $3n$,
- (ii) for $\lfloor \frac{n}{2} \rfloor + 1 \leq \alpha \leq n-1$, that $\{V^{n-\alpha}W^\alpha\} \supset V^{2n-2\alpha}W^{2\alpha-n}$, and the latter has dimension at least $4n - 2\alpha$, by Lemma 2.2.

We also use that $\dim W^n \geq 2$, which can be deduced, for example, from Lemma 2.2, upon choice of a suitable U . Altogether, we get

$$\begin{aligned} w_\ell(n) &= \sum_{\alpha=1}^{\lfloor \frac{n}{2} \rfloor} 3n + \sum_{\alpha=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} (4n - 2\alpha) + 2 \\ &= \sum_{\alpha=1}^{n-1} (4n - 2\alpha) - \sum_{\alpha=1}^{\lfloor \frac{n}{2} \rfloor} (n - 2\alpha) + 2 \\ &= 4n(n-1) - n(n-1) - n\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1) + 2 \\ &= \frac{11}{4}n^2 - \frac{5}{2}n + 2 - \frac{1}{2}\{\frac{n}{2}\}. \end{aligned} \tag{21}$$

We have $c_3 \in J_3^{(2\ell)}$, by Theorem 1.29, and so to calculate G_ℓ , we use

$$\begin{aligned}(w_\ell + 2\tilde{a})(n) &= \frac{11}{4}n^2 - \frac{5}{2}n + 2 - \frac{1}{2}\{\frac{n}{2}\} + (n-1)(n-2) \\ &= \frac{15}{4}n^2 - \frac{11}{2}n + 4 - \frac{1}{2}\{\frac{n}{2}\}.\end{aligned}$$

This gives us

$$\begin{aligned}G_\ell(n) &= \frac{15}{4}\zeta_2(n) - \frac{11}{2}\zeta_1(n) + (4 - \frac{1}{2}\{\frac{n}{2}\})\zeta_0(n) - n(n+1)(n+2) \\ &= \frac{15}{4}\zeta_2(n) - \frac{11}{2}\zeta_1(n) + (4 - \frac{1}{2}\{\frac{n}{2}\})\zeta_0(n) - 3\zeta_2(n) + 3\zeta_1(n) - 2\zeta_0(n) \\ &= \frac{3}{4}\zeta_2(n) - \frac{5}{2}\zeta_1(n) + (2 - \frac{1}{2}\{\frac{n}{2}\})\zeta_0(n).\end{aligned}$$

Thus $G_\ell(2) \geq 0$, and $G_\ell(n) > 0$, for $n \geq 3$, which is what we needed to prove.

In the case that $m > 0$, we use Lemma 2.7, instead of Lemma 2.6. This gives us, for each $\alpha = 1, \dots, n-1$, altogether ℓ instances of k where $\{V^{n-\alpha}W^\alpha\} \subset I_n^{(k)}$, and another ℓ instances of k where $WU^{n-1} \subset I_n^{(k)}$. Using Lemma 1.28, as above, and the fact that $\dim WU^{n-1} = 2$, we get the same estimate (21), which we used above, and hence

$$G_\ell(n) = \frac{3}{4}\zeta_2(n) - \frac{5}{2}\zeta_1(n) + (2 - \frac{1}{2}\{\frac{n}{2}\})\zeta_0(n),$$

as above.

Lemma 2.7 gives us, for each $\beta = 1, \dots, n-1$, also m instances of k for which $\{VW^{n-1-\beta}U^\beta\} \subset I_n^{(k)}$. For such k , we get $\dim I_n^{(k)} \geq 2n+2-2\beta$, for $1 \leq \beta \leq n-2$ (because of $VV = VW + WV$), and 3, for $\beta = n-1$. We also have m instances of U^n , for a contribution of 1. In total,

$$\begin{aligned}w_m(n) &\geq \sum_{\beta=1}^{n-2} (2n+2-2\beta) + 3 + 1 \\ &= n^2 + n - 2.\end{aligned}$$

Hence

$$\begin{aligned}G_m(n) &\geq \zeta_2(n) + \zeta_1(n) - 2\zeta_0(n) - \frac{1}{2}n(n+1)(n+2) \\ &= \zeta_2(n) + \zeta_1(n) - 2\zeta_0(n) - \frac{3}{2}\zeta_2(n) + \frac{3}{2}\zeta_1(n) - \zeta_0(n) \\ &= -\frac{1}{2}\zeta_2(n) + \frac{5}{2}\zeta_1(n) - 3\zeta_0(n).\end{aligned}$$

If we now assume that $\ell \geq m$, then we get

$$\begin{aligned}\frac{1}{m}(\ell G_\ell(n) + m G_m(n)) &\geq \frac{1}{m}(m G_\ell(n) + m G_m(n)) \\ &= G_\ell(n) + G_m(n) \\ &= \frac{1}{4}\zeta_2(n) - (1 + \frac{1}{2}\{\frac{n}{2}\})\zeta_0(n).\end{aligned}$$

This is equal to 0 at $n = 2$, but positive for $n \geq 3$, which is what we needed to prove.

Subcase $m > \ell \geq 0$

We use Lemma 2.8. This gives us, for every $\beta = 1, \dots, n$, the existence of ℓ instances of k where $V^{n-\beta}WU^{\beta-1} \subset I_n^{(k)}$, and m instances of k where $V^{n-\beta}U^\beta \subset I_n^{(k)}$. We estimate these dimensions with Lemma 2.9 to obtain

$$\begin{aligned}
w_\ell(n) &= \sum_{\beta=1}^{\lfloor \frac{n+1}{2} \rfloor} (3n+1-2\beta) + \sum_{\beta=\lfloor \frac{n+1}{2} \rfloor+1}^n (4n+2-4\beta) \\
&= \sum_{\beta=1}^n (4n+2-4\beta) - \sum_{\beta=1}^{\lfloor \frac{n+1}{2} \rfloor} (n+1-2\beta) \\
&= 2n^2 - \lfloor \frac{n+1}{2} \rfloor (\lceil \frac{n+1}{2} \rceil - 1) \\
&= \frac{7}{4}n^2 + \frac{1}{2}\{\frac{n}{2}\}.
\end{aligned} \tag{22}$$

and (not forgetting $\dim U^n = 1$)

$$\begin{aligned}
w_m(n) &= \sum_{\beta=1}^{\lfloor \frac{n}{2} \rfloor} (3n-2\beta) + \sum_{\beta=\lfloor \frac{n}{2} \rfloor+1}^n (4n-4\beta) + 1 \\
&= \sum_{\beta=1}^n (4n-4\beta) - \sum_{\beta=1}^{\lfloor \frac{n}{2} \rfloor} (n-2\beta) + 1 \\
&= 2n^2 - 2n + 1 - \lfloor \frac{n}{2} \rfloor (\lceil \frac{n}{2} \rceil - 1) \\
&= \frac{7}{4}n^2 - \frac{3}{2}n + 1 - \frac{1}{2}\{\frac{n}{2}\}.
\end{aligned}$$

We have

$$\begin{aligned}
(w_\ell + 2\tilde{a})(n) &= \frac{7}{4}n^2 + \frac{1}{2}\{\frac{n}{2}\} + (n-1)(n-2) \\
&= \frac{11}{4}n^2 - 3n + 2 + \frac{1}{2}\{\frac{n}{2}\},
\end{aligned}$$

which gives

$$\begin{aligned}
G_\ell(n) &= \frac{11}{4}\zeta_2(n) - 3\zeta_1(n) + (2 + \frac{1}{2}\{\frac{n}{2}\})\zeta_0(n) - n(n+1)(n+2) \\
&= -\frac{1}{4}\zeta_2(n) + \frac{1}{2}\{\frac{n}{2}\}\zeta_0(n).
\end{aligned}$$

We also have

$$\begin{aligned} G_m(n) &= \frac{7}{4}\zeta_2(n) - \frac{3}{2}\zeta_1(n) + (1 - \frac{1}{2}\{\frac{n}{2}\})\zeta_0(n) - \frac{1}{2}n(n+1)(n+2) \\ &= \frac{1}{4}\zeta_2(n) - \frac{1}{2}\{\frac{n}{2}\}\zeta_0(n). \end{aligned}$$

We observe that for any $n \geq 2$, we have $G_m(n) > 0$. This proves the case where $\ell = 0$. If we now assume that $m > \ell$, then we have, for $n \geq 2$,

$$\ell G_\ell(n) + m G_m(n) > \ell G_\ell(n) + \ell G_m(n) = 0,$$

which proves what we need.

2.6 Case: P smooth fixed point

This is the case where $P = Z(W) \in X^{\text{reg}}$, and $\sigma(P) = P$. In particular, $Z(U) \not\subset X$.

The previous proof of the case $m > \ell \geq 0$, which starts with Formula (22), applies here, too, and so we only need to deal with the case $\ell \geq m \geq 0$. This case is different than before, because we cannot use $VV = VW + WV$ in the estimate for w_ℓ , instead we exploit non-commutativity using Lemma 2.9, to improve the estimate on w_m .

So assume that $\ell \geq m \geq 0$.

We use Lemma 2.6. Thus we have, for every $\alpha = 1, \dots, n$, a total of ℓ instances of k , where $\{V^{n-\alpha}W^\alpha\} \subset I_n^{(k)}$, and a total of m instances of k where $\{W^{n-\alpha}U^\alpha\} \subset I_n^{(k)}$.

We have $\dim V^{n-\alpha}W^\alpha \geq 3n - \alpha$, by Lemma 2.2, even for $\alpha = n$, because the line bundles $L(-P)$ and $L^\sigma(-P)$ are not isomorphic. Therefore, we can take

$$\begin{aligned} w_\ell(n) &= \sum_{\alpha=1}^n (3n - \alpha) \\ &= \frac{5}{2}n^2 - \frac{1}{2}n. \end{aligned}$$

Then

$$\begin{aligned} (w_\ell + 2\tilde{a})(n) &= \frac{5}{2}n^2 - \frac{1}{2}n + (n-1)(n-2) \\ &= \frac{7}{2}n^2 - \frac{7}{2}n + 2, \end{aligned}$$

and

$$\begin{aligned} G_\ell(n) &= \frac{7}{2}\zeta_2(n) - \frac{7}{2}\zeta_1(n) + 2\zeta_0(n) - n(n+1)(n+2) \\ &= \frac{1}{2}\zeta_2(n) - \frac{1}{2}\zeta_1(n). \end{aligned}$$

We observe that this is positive, for all $n \geq 2$, proving the $m = 0$ case.

Now assume that $m > 0$. For $\beta \leq \frac{n}{2}$ we have $\dim\{W^{n-\beta}U^\beta\} \geq 2n - \beta$, and for $\beta \geq \frac{n}{2}$, we have $\dim\{W^{n-\beta}U^\beta\} \geq 3n - 3\beta$, by Lemma 2.9. Therefore, (not forgetting that $\dim U^n = 1$)

$$\begin{aligned} w_m(n) &= \sum_{\beta=1}^{\lfloor \frac{n}{2} \rfloor} (2n - \beta) + \sum_{\beta=\lfloor \frac{n}{2} \rfloor + 1}^n (3n - 3\beta) + 1 \\ &= \sum_{\beta=1}^n (3n - 3\beta) - \sum_{\beta=1}^{\lfloor \frac{n}{2} \rfloor} (n - 2\beta) + 1 \\ &= \frac{3}{2}n^2 - \frac{3}{2}n + 1 - \lfloor \frac{n}{2} \rfloor (\lceil \frac{n}{2} \rceil - 1) \\ &= \frac{5}{4}n^2 - n + 1 - \frac{1}{2}\left\{\frac{n}{2}\right\}. \end{aligned}$$

Hence

$$\begin{aligned} G_m(n) &= \frac{5}{4}\zeta_2(n) - \zeta_1(n) + (1 - \frac{1}{2}\left\{\frac{n}{2}\right\})\zeta_0(n) - \frac{1}{2}n(n+1)(n+2) \\ &= -\frac{1}{4}\zeta_2(n) + \frac{1}{2}\zeta_1(n) - \frac{1}{2}\left\{\frac{n}{2}\right\}\zeta_0(n). \end{aligned}$$

As we are in the case $\ell \geq m$, we get

$$\begin{aligned} \frac{1}{m}(\ell G_\ell(n) + m G_m(n)) &\geq G_\ell(n) + G_m(n) \\ &= \frac{1}{4}\zeta_2(n) - \frac{1}{2}\left\{\frac{n}{2}\right\}\zeta_0(n), \end{aligned}$$

which we already determined to be positive, for all $n \geq 2$.

2.7 Case: P node

Now let us deal with the case where $P = Z(W)$ is a node of X . We can still use the arguments from the above subcase $m > \ell \geq 0$, which starts with Equation (22), and consider this case proved.

So assume that $\ell \geq m \geq 0$.

We remark that P is not fixed by σ , so we still have the convenient fact $VV = VW + WV$. On the other hand, the estimate for the dimension of $V^{n-\alpha-\beta}W^\alpha U^\beta$ drops, but this is made up for by the fact that $c_3 \in I_3^{(3\ell)}$, by Proposition 1.30.

We use Lemma 2.6. Using Lemma 2.10, below, this gives the estimate

$$\begin{aligned}
\overline{w}_\ell(n) &= \sum_{\alpha=1}^{\lfloor \frac{n}{2} \rfloor} 3n + \sum_{\alpha=\lfloor \frac{n}{2} \rfloor+1}^n (5n+1-4\alpha) \\
&= \sum_{\alpha=1}^n (5n+1-4\alpha) - \sum_{\alpha=1}^{\lfloor \frac{n}{2} \rfloor} (2n+1-4\alpha) \\
&= 3n^2 - n - \lfloor \frac{n}{2} \rfloor (2\lceil \frac{n}{2} \rceil - 1) \\
&= \frac{5}{2}n^2 - \frac{1}{2}n.
\end{aligned}$$

But note that this estimate is not optimal. For example, we counted the contribution of W^n with $n+1$, but in fact, it is larger than that, because we cannot have only one node involved. It will be sufficient to improve $w_\ell(n)$ by adding 1 to it:

$$w_\ell(n) = \frac{5}{2}n^2 - \frac{1}{2}n + 1.$$

We have $c_3 \in J_3^{(3\ell)}$, and hence we compute

$$\begin{aligned}
(w_\ell + 3\tilde{a})(n) &= \frac{5}{2}n^2 - \frac{1}{2}n + \frac{3}{2}(n-1)(n-2) + 1 \\
&= 4n^2 - 5n + 3 + 1.
\end{aligned}$$

This gives us

$$\begin{aligned}
G_\ell(n) &= 4\zeta_2(n) - 5\zeta_1(n) + 3\zeta_0(n) - n(n+1)(n+2) + \zeta_0(n) \\
&= \zeta_2(n) - 2\zeta_1(n) + \zeta_0(n) + \zeta_0(n) \\
&= \frac{1}{6}n(2n^2 + 3n - 3) - \{\frac{n}{3}\} + \zeta_0(n).
\end{aligned}$$

This is positive, for all $n \geq 2$, proving the stability bound in the case $m = 0$.

For the case $m > 0$, we note that Lemma 2.6 also gives us

$$\begin{aligned}
w_m(n) &= \sum_{\beta=1}^n (n+1-\beta) \\
&= \frac{1}{2}n^2 + \frac{1}{2}n,
\end{aligned}$$

and hence

$$\begin{aligned}
G_m(n) &= \frac{1}{2}\zeta_2(n) + \frac{1}{2}\zeta_1(n) - \frac{1}{2}n(n+1)(n+2) \\
&= -\zeta_2(n) + 2\zeta_1(n) - \zeta_0(n).
\end{aligned}$$

Then we have

$$\begin{aligned} \frac{1}{m}(\ell G_\ell(n) + m G_m(n)) &\geq G_\ell(n) + G_m(n) \\ &= \zeta_0(n) \\ &> 0. \end{aligned}$$

2.8 Case: Y linear component of X

We assume now that $Y = Z(U) \subset X$. Then Y is a linear component of X , and X is a Neron triangle. Moreover, the automorphism σ acts transitively on the three edges of X . Any choice of non-singular base point of X turns X^{reg} , the non-singular part of X , into a commutative group scheme, such that σ becomes a translation. The point $P = Z(W)$ necessarily lies on X , but is not fixed by σ (of course, it may be a node).

To formulate dimension estimates, let $Y' = \sigma^{-1}Y$, and $Y'' = \sigma^{-2}Y$ be the other two edges of X . Let $Q = Y \cap Y'$, $Q' = Y' \cap Y''$ and $Q'' = Y'' \cap Y$ be the three nodes of X . Let M be a line bundle on X , whose degrees on the three edges, d, d', d'' , are all non-negative. Denote by $\Gamma(M(-\beta Y)) = \Gamma(M \otimes \mathcal{I}_Y^\beta) \subset \Gamma(M)$ the subspace of sections which vanish to order at least β on Y , and $\Gamma(M(-\alpha Q)) = \Gamma(M \otimes \mathfrak{m}_Q^\alpha) \subset \Gamma(M)$ the subspace of sections vanishing to order at least α at Q . (The ideal sheaf of the closed subscheme $Y \subset X$ is locally principal, generated by s , for any non-zero element $s \in U$.)

Lemma 2.10 *We have*

- (i) if $2\beta \leq d' + d'' + 1$, then $\dim \Gamma(M(-\beta Y)) = d' + d'' + 1 - 2\beta$,
- (ii) if $\beta + \gamma \leq d'' + 1$, then $\dim \Gamma(M(-\beta Y - \gamma Y')) = d'' + 1 - \beta - \gamma$,
- (iii) if $2\beta \leq d + d' + d'' + 1$, then $\dim \Gamma(M(-\beta Q)) = d + d' + d'' + 1 - 2\beta$,
- (iv) if $\beta + \gamma \leq d' + 1$, and $\beta + \gamma \leq d'' + 1$, then $\dim \Gamma(M(-\beta Y - \gamma Q')) = d' + d'' + 2 - 2\beta - 2\gamma$,
- (v) if $\beta + \gamma \leq d' + 1$, and $\beta + \gamma \leq d + d'' + 1$, then $\dim \Gamma(M(-\beta Q - \gamma Q')) = d + d' + d'' + 2 - 2\beta - 2\gamma$,
- (vi) if $\alpha + \beta \leq d' + 1$, and $\beta + \gamma \leq d'' + 1$, and $\alpha + \gamma \leq d + 1$, then $\dim \Gamma(M(-\alpha Q - \beta Q' - \gamma Q'')) = d + d' + d'' + 3 - 2\alpha - 2\beta - 2\gamma$.

In all cases, it is important, that $\alpha, \beta, \gamma \geq 1$.

PROOF. These formulas follow easily by breaking up X into rational nodal curves. \square

Corollary 2.11 *For $1 \leq \beta \leq \frac{n+1}{2}$, we have*

$$\Gamma(L_n(-\beta Y)) + \Gamma(L_n(-\beta Y')) = \Gamma(L_n(-\beta Q)),$$

and

$$\Gamma(L_n(-\beta Q)) + \Gamma(L_n(-\beta Y'')) = \Gamma(L_n),$$

and hence also

$$\Gamma(L_n(-\beta Y)) + \Gamma(L_n(-\beta Y')) + \Gamma(L_n(-\beta Y'')) = \Gamma(L_n).$$

Corollary 2.12 *If $1 \leq \gamma \leq \alpha \leq \frac{n+1}{2}$, we have*

$$\Gamma(L_n(-\alpha Y - \gamma Y')) + \Gamma(L_n(-\alpha Y - \gamma Y'')) = \Gamma(L_n(-\alpha Y - \gamma Q')),$$

and

$$\Gamma(L_n(-\alpha Y - \gamma Q')) + \Gamma(L_n(-\alpha Y' - \gamma Q'')) = \Gamma(L_n(-\alpha Q - \gamma Q' - \gamma Q'')),$$

and also

$$\Gamma(L_n(-\alpha Q - \gamma Q' - \gamma Q'')) + \Gamma(L_n(-\alpha Y'' - \gamma Q)) = \Gamma(L_n(-\gamma Q - \gamma Q' - \gamma Q'')).$$

Corollary 2.13 *For $1 \leq \beta \leq \lfloor \frac{n}{3} \rfloor$, we have that*

$$\{V^{n-\beta}U^\beta\} = B_n.$$

For $\lfloor \frac{n}{3} \rfloor + 1 \leq \beta \leq 2\lfloor \frac{n}{3} \rfloor$, we have

$$\{V^{n-\beta}U^\beta\} \supset \Gamma(L_n(-iQ - iQ' - iQ'')),$$

where $i = \beta - \lfloor \frac{n}{3} \rfloor$.

If $n \equiv 1 \pmod{3}$, and $\beta = 2\lfloor \frac{n}{3} \rfloor + 1$, we have

$$\{V^{n-\beta}U^\beta\} \supset \Gamma(L_n(-(\lfloor \frac{n}{3} \rfloor + 1)Y - \lfloor \frac{n}{3} \rfloor Q')),$$

if $n \equiv 2 \pmod{3}$, and $\beta = 2\lfloor \frac{n}{3} \rfloor + 1$, we have

$$\{V^{n-\beta}U^\beta\} \supset \Gamma(L_n(-(\lfloor \frac{n}{3} \rfloor + 1)Q - \lfloor \frac{n}{3} \rfloor Q' - \lfloor \frac{n}{3} \rfloor Q'')), \quad (23)$$

and if $\beta = 2\lfloor \frac{n}{3} \rfloor + 2$, we have

$$\{V^{n-\beta}U^\beta\} \supset \Gamma(L_n(-(\lfloor \frac{n}{3} \rfloor + 1)Y - (\lfloor \frac{n}{3} \rfloor + 1)Y')).$$

PROOF. Note that

$$\underbrace{(UVV) \dots (UVV)}_{\beta} \underbrace{V \dots V}_{n-3\beta} = \Gamma(L_n(-\beta Y)),$$

and

$$\underbrace{VUV \dots VUV}_{\beta} \underbrace{V \dots V}_{n-3\beta} = \Gamma(L_n(-\beta Y')),$$

and

$$\underbrace{VVU \dots VVU}_{\beta} \underbrace{V \dots V}_{n-3\beta} = \Gamma(L_n(-\beta Y'')).$$

The first claim now follows from Corollary 2.11.

The second claim follows from Corollary 2.12 upon considering

$$\underbrace{UUV \dots UUV}_{\beta - \lfloor \frac{n}{3} \rfloor} \underbrace{UVV \dots UVV}_{2\lfloor \frac{n}{3} \rfloor - \beta} \underbrace{V \dots V}_{n - 3\lfloor \frac{n}{3} \rfloor}$$

and its 5 ‘cousins’.

The last three claims follow by similar considerations. \square

Subcase: $\ell = 0$.

To prove the stability estimates, let us start with the case that $\ell = 0$. From Lemma 2.8, we obtain, for every $\beta = 1, \dots, n$ exactly m instances of k where $\{V^{n-\beta}U^\beta\} \subset I_n^{(k)}$. By Corollary 2.13, we have (for $n \geq 1$):

$$\begin{aligned}
w_m(n) &= \sum_{\beta=1}^{\lfloor \frac{n}{3} \rfloor} 3n + \sum_{\beta=\lfloor \frac{n}{3} \rfloor+1}^{2\lfloor \frac{n}{3} \rfloor} (3n+3-6(\beta-\lfloor \frac{n}{3} \rfloor)) \\
&= \sum_{\beta=1}^{\lfloor \frac{n}{3} \rfloor} (6n+3-6\beta) \\
&= (6n+3)\lfloor \frac{n}{3} \rfloor - 3\lfloor \frac{n}{3} \rfloor(\lfloor \frac{n}{3} \rfloor+1) \\
&= \lfloor \frac{n}{3} \rfloor(6n-3\lfloor \frac{n}{3} \rfloor) \\
&= \frac{1}{3}(5n+3\{\frac{n}{3}\})(n-3\{\frac{n}{3}\}) \\
&= \frac{1}{3} \cdot \begin{cases} 5n^2 & \text{if } n \equiv 0 \pmod{3} \\ 5n^2 - 4n - 1 & \text{if } n \equiv 1 \pmod{3} \\ 5n^2 - 8n - 4 & \text{if } n \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

But we have not been optimal, yet, if $n \equiv 1(3)$ or $n \equiv 2(3)$. By Corollary 2.13, if $n \equiv 1(3)$, and $n \geq 4$, we can add $\frac{1}{3}(2n+4)$, but for $n = 1$, we can only add 1 and not 2. In order to have uniform formulas, we will therefore add only $\frac{1}{3}(2n+1)$.

If $n \equiv 2(3)$, and $n \geq 5$, we can add $n+5$ and $\frac{1}{3}(n+1)$, but if $n = 2$, we can only add 5 and 1, and not 7 and 1. Again, in the interest of uniform formulas, we add only $n+3$ and $\frac{1}{3}(n+1)$, for a total of $\frac{2}{3}(2n+5)$.

In sum, we have

$$w_m(n) = \frac{1}{3} \cdot \begin{cases} 5n^2 & \text{if } n \equiv 0 \pmod{3} \\ 5n^2 - 2n & \text{if } n \equiv 1 \pmod{3} \\ 5n^2 - 4n + 6 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Hence,

$$\begin{aligned}
G_m(n) &= \frac{1}{3} \cdot \begin{cases} 5\zeta_2(n) - \frac{3}{2}n(n+1)(n+2) & \text{if } n \equiv 0 \pmod{3} \\ 5\zeta_2(n) - 2\zeta_1(n) - \frac{3}{2}n(n+1)(n+2) & \text{if } n \equiv 1 \pmod{3} \\ 5\zeta_2(n) - 4\zeta_1(n) + 6\zeta_0(n) - \frac{3}{2}n(n+1)(n+2) & \text{if } n \equiv 2 \pmod{3}. \end{cases} \\
&= \frac{1}{6} \begin{cases} \zeta_2(n) + 9\zeta_1(n) - 6\zeta_0(n) & \text{if } n \equiv 0 \pmod{3} \\ \zeta_2(n) + 5\zeta_1(n) - 6\zeta_0(n) & \text{if } n \equiv 1 \pmod{3} \\ \zeta_2(n) + \zeta_1(n) + 6\zeta_0(n) & \text{if } n \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

A direct calculation, using the formulas of Lemma 1.10, shows that $mG_m(n) > 0$, for all $n \geq 2$.

Subcase: $\ell > 0$.

Let us now deal with the case where $\ell > 0$. Lemma 2.8 gives us, for every $\beta = 0, \dots, n-1$ precisely m instances of k where $\{V^{n-\beta-1}WU^\beta\} \subset I_n^{(k)}$. Using the two facts that $VV = WV + VW$ and $WUV + VUW = VUV$, we can replace the single factor of W by a factor of V , and still use the same arguments as before, in the following cases:

- (i) $0 \leq \beta \leq \lfloor \frac{n}{3} \rfloor$,
- (ii) $\lfloor \frac{n}{3} \rfloor + 1 \leq \beta \leq 2\lfloor \frac{n}{3} \rfloor - 1$, or $n \equiv 2 \pmod{3}$.

We get

$$\begin{aligned}
w_\ell(n) &\geq \sum_{\beta=0}^{\lfloor \frac{n}{3} \rfloor} 3n + \sum_{\beta=\lfloor \frac{n}{3} \rfloor+1}^{2\lfloor \frac{n}{3} \rfloor-1} (3n+3-6(\beta-\lfloor \frac{n}{3} \rfloor)) \\
&= \sum_{\beta=1}^{\lfloor \frac{n}{3} \rfloor} (6n+3-6\beta) + 3n - (3n+3-6\lfloor \frac{n}{3} \rfloor) \\
&= \lfloor \frac{n}{3} \rfloor (6n-3\lfloor \frac{n}{3} \rfloor) + 6\lfloor \frac{n}{3} \rfloor - 3 \\
&= \frac{1}{3} \cdot \begin{cases} 5n^2 + 6n - 9 & \text{if } n \equiv 0 \pmod{3} \\ 5n^2 + 2n - 16 & \text{if } n \equiv 1 \pmod{3} \\ 5n^2 - 2n - 25 & \text{if } n \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

If $n \equiv 2 \pmod{3}$, we can add a summand of $3n+3-6\lfloor \frac{n}{3} \rfloor$, and we get instead

$$w_\ell(n) = \frac{1}{3} \cdot \begin{cases} 5n^2 + 6n - 9 & \text{if } n \equiv 0 \pmod{3} \\ 5n^2 + 2n - 16 & \text{if } n \equiv 1 \pmod{3} \\ 5n^2 + n - 4 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

These estimates are still not sufficient.

For $n \equiv 1 \pmod{3}$, we consider $\{V^{n-\beta-1}WU^\beta\}$ with $\beta = 2\lfloor \frac{n}{3} \rfloor$. It contains the three subspaces

$$\underbrace{UVV \dots UVV}_{\lfloor \frac{n}{3} \rfloor} W, \quad \underbrace{UVU \dots UVU}_{\lfloor \frac{n}{3} \rfloor} W, \quad \underbrace{VUU \dots VUU}_{\lfloor \frac{n}{3} \rfloor} W.$$

Considering the first two, we see that the factor W can be moved into a position where it does not affect the dimension calculation. This means that we get a dimension estimate for the sum of these three spaces which is worse by 1, than the dimension estimate $3n+3-6(\beta-\lfloor \frac{n}{3} \rfloor)$, which we used for $\{V^{n-\beta}U^\beta\}$. We get an estimate of $3n+2-6\lfloor \frac{n}{3} \rfloor = n+4$.

For $n \equiv 2 \pmod{3}$, consider the contribution (23), which gave us a dimension estimate of $n+3$, for $n \geq 2$. In the presence of a factor of W , this estimate drops by 2 to $n+1$.

This gives us the following improved formulas for w_ℓ :

$$w_\ell(n) = \frac{1}{3} \cdot \begin{cases} 5n^2 + 6n - 9 & \text{if } n \equiv 0 \pmod{3} \\ 5n^2 + 5n - 4 & \text{if } n \equiv 1 \pmod{3} \\ 5n^2 + 4n - 1 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

We have that $c_3 \in J_3^{(3\ell)}$, by Proposition 1.30, and so we add $\frac{3}{2}(n-1)(n-2)$ to get

$$(w_\ell + 3\tilde{a})(n) = \frac{1}{6} \cdot \begin{cases} 19n^2 - 15n & \text{if } n \equiv 0 \pmod{3} \\ 19n^2 - 17n + 10 & \text{if } n \equiv 1 \pmod{3} \\ 19n^2 - 19n + 16 & \text{if } n \equiv 2 \pmod{3}, \end{cases}$$

which gives, by subtracting $3\zeta_2 - 3\zeta_1 + 2\zeta_0$,

$$G_\ell(n) = \frac{1}{6} \cdot \begin{cases} \zeta_2(n) + 3\zeta_1(n) - 12\zeta_0(n) & \text{if } n \equiv 0 \pmod{3} \\ \zeta_2(n) + \zeta_1(n) - 2\zeta_0(n) & \text{if } n \equiv 1 \pmod{3} \\ \zeta_2(n) - \zeta_1(n) + 4\zeta_0(n) & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

A direct calculation shows that $\ell G_\ell(n) > 0$, for all $n \geq 2$. This proves the required stability estimate for $m = 0$.

If both $\ell > 0$, and $m > 0$, Lemma 2.8 shows that we may simply add our values for ℓG_ℓ and $m G_m$ to prove the stability estimate. Since they are positive individually, this has already been observed.

3 Moduli stacks

3.1 Moduli stacks of stable regular algebras

Recall that we are working over an algebraically closed field \mathbb{C} , of characteristic not equal to 2 or 3.

Definition 3.1 A **flat family of stable regular algebras**, parametrized by the \mathbb{C} -scheme T , is a graded vector bundle $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$ over T , endowed with the structure of sheaf of graded \mathcal{O}_X -algebras, such that for every $t \in T$, the fibre \mathcal{A}_t is a stable regular algebra as in Theorem 1.2.

Let us denote the \mathbb{C} -stack of families of stable regular algebras by $\mathfrak{M}^{s,r}$.

Theorem 3.2 *The stack $\mathfrak{M}^{s,r}$ of flat families of stable regular algebras is a smooth Deligne-Mumford stack of finite type. It has 4 components, $\mathfrak{M}_A^{s,r}$, $\mathfrak{M}_B^{s,r}$, $\mathfrak{M}_E^{s,r}$, and $\mathfrak{M}_H^{s,r}$. Concretely,*

- (A) $\mathfrak{M}_A^{s,r} = [U/G_{216}]$, where $U \subset \mathbb{P}^2$ is the complement of 4 concurrent lines, and 4 points in \mathbb{P}^2 . The group G_{216} is the group of automorphisms of the oriented affine plane over \mathbb{F}_3 (which has 216 elements). It acts U via its quotient $SL_2(\mathbb{F}_3)$,

(B) $\mathfrak{M}_B^{s,r} = [V/\mathbb{Z}_4]$, where $V \subset \mathbb{P}^1$ is the complement of 3 points in \mathbb{P}^1 , and \mathbb{Z}_4 acts via its quotient \mathbb{Z}_2 .

(E) $\mathfrak{M}_E^{s,r} = B\mathbb{Z}_3$,

(H) $\mathfrak{M}_H^{s,r} = B\mathbb{Z}_4$.

PROOF. Let \mathcal{A} be a flat family of stable regular algebras parametrized by the scheme T . As all members of \mathcal{A} are elliptic, the triple (X, σ, L) of \mathcal{A} is a flat family of elliptic triples. We have a short exact sequence of vector bundles on T :

$$0 \longrightarrow R \longrightarrow V \otimes V \longrightarrow \pi_*(L \otimes L^\sigma) = \mathcal{A}_2 \longrightarrow 0 .$$

Here $\pi : X \rightarrow T$ is the structure morphism. The algebra \mathcal{A} is recovered from $R \rightarrow V \otimes_{\mathcal{O}_T} V$ as the quotient of the tensor algebra on V divided by the two sided sheaf of ideals generated by R .

Now $\sigma : X \rightarrow X$ induces a group automorphism $\text{Pic}^0(\sigma) : \text{Pic}(X/T)^0 \rightarrow \text{Pic}(X/T)^0$. The order of $\text{Pic}^0(\sigma)$ is necessarily finite, of order 1, 2, 3, or 4, as we are avoiding characteristic 2 and 3. The order of $\text{Pic}^0(\sigma)$ is also locally constant over T , and so T breaks up into 4 open and closed subschemes T_i , where $T_i \subset T$ is the locus where the order of $\text{Pic}^0(\sigma)$ is i . This proves that $\mathfrak{M}^{r,s}$ also breaks up into 4 open and closed substacks $\mathfrak{M}_i^{r,s}$.

Let us deal with each of the 4 components in turn, and assume that the order of $\text{Pic}^0(\sigma)$ is constant.

Case A. First, assume that the order of $\text{Pic}^0(\sigma)$ is 1, so that $\text{Pic}^0(\sigma)$ is the identity of $\text{Pic}^0(X/T)$. By our classification of stable algebras, the automorphism σ acts transitively on the set of components in every fibre. Therefore, by Theorem II 3.2 in [5], for any section P of X^{reg} , there exists a unique structure of generalized elliptic curve on X , having P as origin, such that σ acts as translation. In particular, X^{reg} is a commutative group scheme, which acts on X , and there is another section S of X^{reg} , such that $\sigma(Q) = Q + S$, for all sections Q of X .

Using this, we prove that in the the fibered product

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & T \\ \downarrow & & \downarrow L \\ X & \xrightarrow{Q \mapsto \mathcal{O}(Q + \sigma Q + \sigma^{-1} Q)} & \text{Pic}^3(X) \end{array}$$

the scheme Z is a form of the oriented affine plane over \mathbb{F}_3 . In particular, Z is a finite étale cover $Z \rightarrow T$ of degree 9.

Then, at least étale locally, we can assume that P is a section of Z , which we use to turn X into a generalized elliptic curve. then Z is equal to the scheme of 3-division points in X , in particular, Z is an oriented vector bundle over \mathbb{F}_3 . We can choose, étale locally, an oriented basis Q_1, Q_2 for Z . We let S be the section of X^{reg} , such that σ is translation by S . From the fact that the algebra \mathcal{A} is elliptic, it follows that S avoids $Z \subset X$.

Thus, at least étale locally, we can associate to \mathcal{A} a generalized elliptic curve with full oriented level-3-structure, with an extra point S on it. The ambiguity is in the choice of the oriented coordinate system (P, Q_1, Q_2) for the bundle of affine planes Z . The only restriction is that in singular fibres, S avoids the 3-division sections, that S stays away from X^{sing} , and that in the singular fibres, S avoids the connected component of P .

Thus, conversely, let $\mathfrak{M}(3)$ be the moduli scheme of generalized elliptic curves with full oriented level-3-structure. Let $\mathfrak{E}(3) \rightarrow \mathfrak{M}(3)$ be the universal curve. It classifies quintuples (E, P, Q_1, Q_2, S) , where (E, P) is a generalized elliptic curve, whose geometric fibres are smooth or triangles, Q_1 and Q_2 are 3-division points on E , forming an oriented bases for E_3 , and $S \in E$ is a further point on E . Let $\mathfrak{E}(3)^0 \subset \mathfrak{E}(3)$ be the open subscheme defined by the conditions that S is not a node of E , not in the component of identity in any triangle, and not in E_3 . Using the Hesse family of elliptic curves, it is not hard to identify $\mathfrak{E}(3)_0$ with the complement of 4 concurrent lines and 4 points in \mathbb{P}^2 .

The scheme $\mathfrak{E}(3)_0$ has the tautological triple $(E, \tau_S), \mathcal{O}(3P))$ over it. Associated to this triple is a flat family of stable regular algebras $\mathcal{A}_{\mathfrak{E}(3)_0}$, parametrized by $\mathfrak{E}(3)_0$. The scheme of isomorphisms of $\mathcal{A}_{\mathfrak{E}(3)_0}$ is canonically identified with the transformation groupoid of G_{216} acting on $\mathfrak{E}(3)_0$.

We have seen above, that the algebra $\mathcal{A}_{\mathfrak{E}(3)_0}$ is versal, i.e., every flat family of stable regular algebras is étale locally induced from $\mathcal{A}_{\mathfrak{E}(3)_0}$. This finishes the proof that $\mathfrak{M}_1^{s,r} \cong [\mathfrak{E}(3)_0/G_{216}]$.

Case B. Let $\mathfrak{M}(2)_0$ be the moduli stack of (non-singular) elliptic curves with a fixed 2-division point. It classifies triples (E, P, Q) , where (E, P) is a smooth elliptic curve, and Q is a non-zero section of E_2 . Note that the Legendre family of elliptic curves identifies $\mathfrak{M}(2)_0$ with $[V/\mathbb{Z}_4]$, where $V \subset \mathbb{P}^1$ is the complement of 3 points in \mathbb{P}^1 . To (E, P, Q) we associate the elliptic triple (X, σ, L) , given by $X = E$, $\sigma : X \rightarrow X$ is the involution $R \mapsto Q - R$, whose fixed points are the 4 second roots of Q , and $L = \mathcal{O}(2P + Q)$. The associated family of algebras is a flat family of stable regular algebras of Type B over $\mathfrak{M}(2)_0$.

Conversely, let $\mathcal{A} \rightarrow T$ be a flat family of elliptic regular algebras, with associated flat family of elliptic triples (X, σ, L) . Assume that the order of $\text{Pic}^0(\sigma)$ is 2. Then $\text{Pic}^0(\sigma)$ is necessarily the multiplication by -1 automorphism. This implies that for any two local sections Q, Q' of X , the line bundles $\mathcal{O}_X(Q + \sigma(Q))$ and $\mathcal{O}_X(Q' + \sigma(Q'))$ are isomorphic. Thus there exists a unique, and therefore global, section P , such that $\mathcal{O}(Q + \sigma(Q) + P) \cong L$, for any Q . This makes (X, P) into an elliptic curve, and the automorphism σ into $R \mapsto \sigma(P) - R$. Regularity of the triple (X, σ, L) implies that $Q = \sigma(P)$ is a 2-division point on (X, P) . We see that $\mathfrak{M}_2^{s,r} \cong \mathfrak{M}(2)_0$.

Cases E and H. Left to the reader. \square

3.2 Density

Proposition 3.3 *Suppose that \mathcal{A} is a flat family of graded q -truncated algebras, parametrized by the finite type k -scheme T . Then the locus of points $t \in T$, such that the fibre \mathcal{A}_t is the q -truncation of a stable regular algebra is open in T .*

PROOF. By definition,

$$\mathcal{A} = \bigoplus_{n=0}^q \mathcal{A}_n,$$

is a direct sum of vector bundles, where $\mathcal{A}_0 = \mathcal{O}_T$. Let us assume that t_0 is a point of T , such that the fibre $\mathcal{A}|_{t_0}$ of \mathcal{A} over t_0 is the truncation of a non-singular elliptic regular algebra. We will prove that there exists an open neighbourhood U of t_0 in T , such that for every $t \in U$, the fibre $\mathcal{A}|_t$ is a non-singular elliptic regular algebra as well.

The rank of the vector bundle \mathcal{A}_n is a locally constant function on T , and so by restricting to an open neighbourhood of t_0 , we may assume that it is constant. It is then equal to $\frac{1}{2}(n+1)(n+2)$, because it takes that value at t_0 , by Formula (1.15) in [2].

Let us write $V = \mathcal{A}_1$. This is now a vector bundle of rank 3 on T . Multiplication in \mathcal{A} defines a homomorphism of vector bundles

$$V \otimes_{\mathcal{O}_T} V \rightarrow \mathcal{A}_2. \quad (24)$$

The locus of points in T , over which (24) is not surjective is closed in T . Over the point t_0 , the homomorphism (24) is surjective, because $\mathcal{A}|_{t_0}$ is generated in degree 1. So the locus of points in T where (24) is surjective is an open neighbourhood of t_0 , and by restricting to this open neighbourhood, we may assume that (24) is surjective. Then the kernel of (24) is a vector bundle R of rank 3 on T .

By the same reasoning, we may assume that $\mathcal{A}_2 \otimes_{\mathcal{O}_T} V \rightarrow \mathcal{A}_3$ and $V \otimes_{\mathcal{O}_T} \mathcal{A}_2 \rightarrow \mathcal{A}_3$ are epimorphisms of vector bundles, and that the respective kernels K and K' are vector bundles as well. Both vector bundles K and K' have rank 8.

We have the following commutative diagram of coherent sheaves of \mathcal{O}_T -modules:

$$\begin{array}{ccccccc} W & \xrightarrow{\beta} & R \otimes V & \longrightarrow & K' & \longrightarrow & Q' \\ \alpha \downarrow & & \downarrow & & \downarrow & & \\ V \otimes R & \longrightarrow & V \otimes V \otimes V & \longrightarrow & V \otimes \mathcal{A}_2 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ K & \longrightarrow & \mathcal{A}_2 \otimes V & \longrightarrow & \mathcal{A}_3 & & \\ \downarrow & & & & & & \\ Q & & & & & & \end{array}$$

All tensor products are over \mathcal{O}_T . All rows and columns are exact, when completed with zeros on all ends. By the snake lemma, $Q' = Q$. Since $\mathcal{A}|_{t_0}$ is quadratic, the homomorphisms $V \otimes R \rightarrow K$ and $R \otimes V \rightarrow K'$ are surjective near t_0 , and so we can assume they are surjective, and that $Q' = Q = 0$. Then all sheaves in our diagram are locally free, and in particular W is a vector bundle of rank 1.

We may also assume that the images of α and β have full rank, i.e., any non-zero section of ω of W induces isomorphisms $\beta(\omega)^* : R^* \rightarrow V$ and $\alpha(\omega)^* : V^* \rightarrow R$, because this is the case at the point t_0 .

We now consider the projective bundle $\mathbb{P}(V) \rightarrow T$ of one-dimensional quotients of V . From the homomorphism of vector bundles $R \rightarrow V \otimes V$, we obtain homomorphisms

$$\Lambda^3 R \longrightarrow \Lambda^3 V \otimes_{\mathcal{O}_T} \mathrm{Sym}^3 V \quad \text{and} \quad \Lambda^3 R \longrightarrow \mathrm{Sym}^3 V \otimes_{\mathcal{O}_T} \Lambda^3 V,$$

and hence

$$\Lambda^3 V^* \otimes \Lambda^3 R \longrightarrow \mathrm{Sym}^3 V \quad \text{and} \quad \Lambda^3 R \otimes \Lambda^3 V^* \longrightarrow \mathrm{Sym}^3 V. \quad (25)$$

The arrows (25) are both strict monomorphisms of vector bundles over T , and hence X_1, X_2 are flat families of Cartier divisors of degree 3 in $\mathbb{P}(V)$. In fact, by our assumption that both α and β have full rank, we have $X_1 = X_2$, i.e., we are in the semi-standard case. We will call this scheme $X = X_1 = X_2$.

Moreover, $R \rightarrow V \otimes V$ defines a family of subschemes Γ in $\mathbb{P}(V) \times_T \mathbb{P}(V)$. By construction the projections factor $\pi_1 : \Gamma \rightarrow X$ and $\pi_2 : \Gamma \rightarrow X$. At t_0 , both these morphisms are isomorphisms, and so by properness of Γ and X , we may assume that they are isomorphisms.

This gives us an elliptic triple, from which we can construct a flat family of elliptic regular algebras \mathcal{A}' , together with a morphism of graded algebras $\mathcal{A}' \rightarrow \mathcal{A}$. At the point t_0 it is an isomorphism, so we may assume that it is an isomorphism.

Let us finish by proving that there is an open neighbourhood of t_0 where the triple (X, σ, L) is regular. First note that the condition of being regular or exceptional is open: it is the locus where $R^1 \pi_*(L \otimes (L^\sigma)^{-\otimes 2} \otimes L^{\sigma^2})$ has rank 1.

If \mathcal{A}_{t_0} is of type B, E, or H, we can pass to the open set where X is smooth, to get rid of the exceptional locus.

So assume that t_0 is of type A. If X is smooth at t_0 , we simply pass to the open neighbourhood where X is smooth. If X is not smooth at t_0 it is a triangle, which is being rotated by σ . We now restrict to the open neighbourhood of t_0 where X is nodal and $\mathrm{Pic}^0(\sigma)$ is the identity.

We also exclude all points where the algebra is not stable, as this locus is closed. Then every fibre is either smooth, or a triangle on which σ acts by rotation, or exceptional, in which case σ acts by swapping components. Thus in every fibre, σ acts transitively on connected components, and by by Lemme II 1.7 of [5], we may endow X with the structure of a generalized elliptic curve. Then by Proposition II 1.15 of [5], the exceptional locus, which is the locus where X has two components, is closed in T , and so we can remove it, and we are left with a family of stable regular algebras, as required. \square

Corollary 3.4 *For every q , the moduli stack of stable algebras \mathfrak{M}_q^s has an open substack isomorphic to*

$$\mathfrak{M}_A^{s,r} \amalg \mathfrak{M}_B^{s,r} \amalg \mathfrak{M}_E^{s,r} \amalg \mathfrak{M}_H^{s,r}.$$

Each stack $\mathfrak{M}_i^{s,r}$, for $i \in \{A, B, E, H\}$, is an open dense substack in the irreducible component of \mathfrak{M}_q^s which contains it.

We can unfortunately not prove that every stable algebra is regular. There may be further stable algebras in the boundary of the non-proper stacks $\mathfrak{M}_A^{s,r}$ and $\mathfrak{M}_B^{s,r}$, or there may be entire components consisting of stable algebras which are not regular.

In particular, it may happen that exceptional algebras are stable, as their triples can satisfy our stability criterion, but we have not examined this possibility closer.

For every q , we have a projective coarse moduli space of S-equivalence classes of semi-stable algebras, but we do not have a complete description of these, except for low values of q .

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